Descriptive Units of Heterogeneity: An Axiomatic Approach to Measuring Heterogeneity

(Latest Version)

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July 2, 2025†

Abstract

This paper addresses the challenge of measuring heterogeneity in systems, noting that existing measures often fall short in providing cardinal interpretations or comparability across contexts. Using an axiomatic framework, I examine the strengths and limitations of inequality and concentration measures and identify general properties that alternative measures of heterogeneity should satisfy. Based on these principles, I propose a new family of measures—Descriptive Units of Heterogeneity (DUH)—which overcome prior limitations without restricting applicability. DUH achieves the generalized comparability of concentration measures while remaining sensitive to changes in the distribution of small groups. Two empirical examples demonstrate the usefulness of DUH in diverse settings, including racial composition in cities and firm-level revenue diversification.

Keywords: Diversity, Concentration, Heterogeneity
JEL: B41, D30, D63, J15, L11

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†I thank Jon Eguia, Ce Liu, Sarah Mirrow, Hanzhe Zhang, and referees for suggestions and feedback.

1 Introduction

Measuring and ordering the heterogeneity of systems, such as the diversity of a categorical outcome observed within populations, has been a longstanding problem with only imperfect solutions. While inequality and concentration measures such as the Gini coefficient and the Herfindal-Hirschman index (HHI) have commonly been repurposed to measure heterogeneity, features desirable for their intended use can prove to be inappropriate in this case (Kvålseth, 2022; Nunes, Trappenberg, and Alda, 2020). I address these challenges by axiomatizing a family of measures that satisfy desiderata better suited for measuring heterogeneity than inequality or concentration. These new measures are termed the Descriptive Units of Heterogeneity (DUH). The departure from the term index emphasizes the measures' cardinal interpretation and its distinction from measures of inequality and concentration.

Let a system be defined as a partition of a finite, non-empty population into disjoint subsets, each associated with a distinct group label. The concept of heterogeneity within such a system pertains to the distribution of group sizes: greater heterogeneity corresponds to more balanced group sizes, whereas homogeneity reflects the dominance of one or a few groups. For instance, a bi-partition of a population into two equally sized groups (e.g., 50–50) is considered more heterogeneous than a highly imbalanced division (e.g., 99–1). Similarly, a tri-partition such as (45, 45, 10) exhibits greater heterogeneity than (85, 5, 10), due to the more equitable distribution of individuals across groups. While intuitive comparisons can often be made between such configurations, ambiguity arises in intermediate cases—for example, comparing (55, 35, 10) with (60, 20, 20). The challenge becomes more pronounced as the number of groups increases, raising the question of how to formally characterize and compare heterogeneity across arbitrary partitions.

Measuring heterogeneity fundamentally involves a dimension-reduction problem. When examining a system comprising numerous groups, the goal is to simplify the complexity while preserving essential information. Selecting appropriate dimensions to characterize heterogeneity is crucial to defining measures capable of effectively detecting changes in the system. A sensible way to proceed is to determine first the bounds of the measure, and then the ordering of systems that are neither perfectly heterogeneous nore perfectly homogeneous. The numerical bounds of any heterogeneity measure are established by designating perfect heterogeneity as any system in which all groups contain an equal number of elements, and perfect homogeneity as any system in which all but one group have zero elements. Subsequent tasks include (1) balancing the relative influence of large groups versus the distribution of elements across small groups, and (2) clearly defining how the number of groups contributes to heterogeneity. These considerations are essential since, for instance, it might seem intuitive that the system $\left(\frac{1}{2}, \frac{1}{2}\right)$ is less heterogeneous than $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$; however, formal criteria are necessary to justify such comparisons.

In applied settings, system heterogeneity is most commonly quantified using either dispersion measures or concentration measures (James and Taeuber, 1985). Dispersion measures, originally devised for the measurement of inequality, typically rely on an implicit ordering of outcomes such as household income. Although these measures are formally invariant under permutations of outcome labels, meaningful interpretation of changes in their values frequently presupposes a commonly accepted ordering of those labels. Concentration measures, by contrast, originated in industrial organization and are primarily sensitive to the dominance of large firms, thus largely neglecting distributional variation among entities with smaller total shares in the market. While both types of measures fall short of providing a comprehensive account of heterogeneity, their underlying logic offers a useful starting point for further refinement.

Dispersion measures, more commonly known as measures of inequality such as the Gini coefficient, quantify inequality with the distance between the observed population distribution and a benchmark distribution. Making inference on system heterogeneity with dispersion measures are simple but it is often limited by the Lorenz criterion. Atkinson (1970) show that comparing any two systems using the distance between distributions requires significant restrictions on the domain of comparable systems; otherwise, different measures can be made to rank any two systems in opposing ways. Within the bounds of such restrictions, the interpretation of a dispersion index is simply "the higher the number, the greater the heterogeneity." One well-known example of a dispersion index is the Gini coefficient. While its simple interpretation contributes to its popularity, Schwartz and Winship (1980) point out that many empirical researchers fail to account for these restrictions when using the Gini coefficient to rank income inequality (pp. 2, 8). Such failure can lead to obscured inferences (pp. 9-13). To relax the Lorenz criterion, Weymark (1981) characterizes a set of generalized Gini absolute and relative inequality measures that rank systems consistently given assigned weights, even when the Lorenz criterion is not satisfied.

Concentration measures measure the richness of information from select subgroups of the population. The usage of these units are not limited by the Lorenz criterion, but they tend to poorly reflect the information provided by the distribution of smaller groups. The Herfindahl-Hirschman index (HHI) and Shannon's entropy (SE) are popular examples of concentration measures. By emphasizing the influence of large groups, concentration measures well reflect changes in heterogeneity in systems when a large group shrinks. However, this feature can cause concentration measures to report negligible changes in heterogeneity when there are drastic changes between small groups. Additionally, because the objective of concentration measures is to measure concentration, they omit any information provided by the presence of groups with zero elements (zero-groups). These features are not desirable for a measure of heterogeneity because the existence of multiple zero-groups increases the variance of group sizes.

I axiomatize a new family of measures that (1) yield cardinal interpretation, (2) shift the focus away from large groups, (3) account for the presence of zero-groups, and (4) share basic properties with existing measures. I adopt a new yet intuitive approach to model heterogeneity with two determinants: (1) contribution from the relative size of the largest group and (2) contribution from the evenness of the rest of the groups. This approach builds on the axiomatic underpinnings of the dispersion and concentration measures and allows for the addition of certain desiderata more suitable for measuring heterogeneity. I divide the axioms into two groups: fundamental axioms and characterization axioms.

Fundamental axioms formalize the most common desiderata across measures of heterogeneity and induce partial ordering. The axioms are group symmetry (GSYM), scale invariance (INV), and the principle of transfers (PT). GSYM and INV state that two systems with the same number of groups are equally heterogeneous if one is a permutation or rescaling of the other. PT establishes how two marginally different systems should be ordered. Most dispersion and concentration measures satisfy all three fundamental axioms.

Characterization axioms induce total ordering and uniquely characterize measures. I define independence (IND), the principle of proportional transfers (PPT), contractibility (CON), and unity (UNI). IND ensures that the influence of the largest group and that of the evenness of minority groups are orthogonal; PPT refines PT with cardinal interpretation; CON requires that a measure of heterogeneity is not invariant to the addition of zero-groups; and UNI normalizes any measure to be between 0 and 1. These axioms characterize a family of measures that focus on rendering the comparison of heterogeneity between systems descriptive—cardinally interpretable, reflective of changes to small groups, and more suitable for comparing the heterogeneity between arbitrary systems. This family of measures is termed the descriptive units of heterogeneity (DUH).

As a direct result of the fundamental axioms, the rank correlation between DUH and other measures are high. However, many pairs of systems have opposing relative orders because DUH are not intended for measuring either dispersion or concentration. A closer look at the distribution of measure values reveal that, given fixed size of the largest group, DUH has a much higher variation in values than the Gini coefficient, HHI, and SE.

An essential aspect of system heterogeneity lies in how a population is partitioned into groups. In particular, meaningful comparisons of heterogeneity between systems require that the partitions be defined over a set of group labels relevant to the context. To illustrate the importance of this caveat, I present an example in which differing group labels yield conflicting conclusions, thereby motivating the notion of a reasonable collection of groups. To further demonstrate the applicability of the DUH, I consider two sets of examples. First, using the evolving racial composition of a hypothetical city, I show that DUH is particularly well-suited for contexts in which the primary concern is heterogeneity itself, rather than inequality or concentration. Second, drawing on changes in Apple's revenue sources from 2012 to 2023, I demonstrate that DUH is sensitive to shifts in the distribution of minority groups and can more effectively capture the growth of small categories than standard measures of inequality or concentration.

The remainder of the paper is organized as follows. Section 2 introduces the foundational primitives for system heterogeneity, reviews existing measures of dispersion and concentration, and motivates the development of a new approach. Section 3 presents the fundamental and characterization axioms and introduces the proposed framework, which identifies two key determinants of heterogeneity. Section 4 defines DUH and compares their behavior to that of inequality and concentration measures. Section 5 outlines best practices for empirical application and provides illustrative examples that demonstrate the advantages of DUH across diverse contexts. Formal proofs establishing the uniqueness of the proposed family of measures under the stated axioms are provided in the Appendix.

2 Model

2.1 Primitives

The objective of a measure of heterogeneity is to induce a total order over the heterogeneity of populations. The following subsection defines notations used for a convenient discussion.

Definition 1. *A population $\mathbb{P}$ is a finite, non-empty collection of elements where $|\mathbb{P}| = N \in \mathbb{N}$. Let $\mathcal{P}$ denote the collection of all $\mathbb{P}$.*

Definition 2. *Let Θ be the universe of distinct group labels. $\theta \in 2^{\Theta} \setminus \varnothing$ is a finite, non-empty collection of group labels where $|\theta| = G \in \mathbb{N}$ and $\theta = \{\theta_1, \theta_2, \ldots, \theta_G\} \subseteq \Theta$.*

Definition 3. *Let $\mathbb{P} \in \mathcal{P}$ be an arbitrary population such that $|\mathbb{P}| = N \in \mathbb{N}$ and $\theta \in 2^{\Theta} \setminus \varnothing$ be an arbitrary collection of group labels such that $|\theta| = G \in \mathbb{N}$. Let $S : \mathcal{P} \times 2^{\Theta} \setminus \varnothing \to \mathbb{Z}_+^G$ be a* *function that partitions $\mathbb{P}$ into G distinct sets according to the elements of θ and outputs a vector $(n_1, n_2, \ldots, n_G)$ where $\forall g \in \{1, \ldots, G\}$, n_g is the number of elements in $\mathbb{P}$ that belongs to the partition associated with the label $\theta_g \in \theta$.*

Definition 4. *A system $S(\mathbb{P}, \theta)$ is the image of the function S with inputs $\mathbb{P}$ and θ. For ease of notation, denote the system $S(\mathbb{P}, \theta)$ as $S_{\mathbb{P}}^{\theta}$.*

Definition 5. *A function $\Phi : \mathbb{Z}_+^G \to \mathbb{R}$ is a measure of heterogeneity if, for any two systems S_1^{θ} and S_2^{ρ},*

$$\Phi(S_2^{\rho}) \geq \Phi(S_1^{\theta}) \iff S_2^{\rho} \text{ is weakly more heterogeneous } (\succsim) \text{ than } S_1^{\theta}.$$

Definition 6. *A system $S_{\mathbb{P}}^{max}$ of G groups is maximally heterogeneous if $\exists\, n \in \mathbb{N}$ such that:*

$$S_{\mathbb{P}}^{max} = (n_1, n_2, \ldots, n_G) = n \cdot (1, 1, \ldots, 1).$$

Definition 7. *A system $S_{\mathbb{P}}^{min}$ of G groups is minimally heterogeneous (or perfectly homogeneous) if all but one element of $S_{\mathbb{P}}^{min}$ is 0. In other words, $S_{\mathbb{P}}^{min} = (0, 0, \cdots, 0, n, 0, \cdots, 0, 0)$.*

Under Definitions 6 and 7, it is convenient to restrict any measure of heterogeneity to a bounded interval, typically between 0 or 1 and a positive real number. Such normalization facilitates the construction of a total order over systems that are neither maximally nor minimally heterogeneous, allowing for arbitrary but principled comparisons subject to reasonable constraints (Marshall, Olkin, and Arnold, 1979).

Although originally developed for other purposes, measures of dispersion and concentration are frequently employed in applied research as measures for heterogeneity (James and Taeuber, 1985). While often appropriate, features that are desirable in their original domains may be less suitable when the objective is to assess heterogeneity. Nonetheless, many of these measures have been axiomatized using desiderata that are foundational to the study of heterogeneity. For example, Rothschild and Stiglitz (1971, 1973) and Schwartz and Winship (1980) and Schwartz and Winship (1980) provided axiomatizations of the Gini coefficient; Weymark (1981) characterized the generalized Gini absolute inequality index;Chakravarty and Eichhorn (1991) and Kvålseth (2022) addressed the Herfindahl–Hirschman Index (HHI); and Shannon's entropy (SE) has been axiomatized by Nambiar, Varma, and Saroch (1992), Suyari (2004), and Chakrabarti, Chakrabarty et al. (2005). The literature surrounding these measures is well-developed, with extensive theoretical justification for their axioms grounded in various approaches to social welfare (Chakravarty, 2015; Cowell, 2000; Lambert, 2002).

2.2 Measures of Dispersion/Inequality

Dispersion measures compare the observed distribution to a benchmark distribution. The Gini coefficient, a well-known dispersion measure, regards a system as the rank distribution of a single outcome, typically household income, and compares that distribution to the uniform distribution. The distance between the two distributions represents the amount of heterogeneity of the system.

Blackorby and Donaldson (1978) shows that the Gini coefficient can be written as:

$$Gini(S_{\mathbb{P}}^{\theta}) = 1 - \frac{1}{G^2} \cdot \left(\frac{\sum_{g=1}^{G} (2g-1)\hat{n}_g}{\mu(n)} \right),$$

where G is the number of groups, $\mu(n) = \frac{1}{G}\sum_{g=1}^{G} n_g$, and $\hat{n}_g$ is the size of group g in the ordered permutation of $\hat{S}_{\mathbb{P}}^{\theta}$ such that $\hat{n}_1 \geq \hat{n}_2 \geq \cdots \geq \hat{n}_G$. The Gini coefficient evaluates inequality through the relationship: "the top $x\%$ of households earn the top $y\%$ of income." Under perfect equality—corresponding to maximal heterogeneity in group sizes—this relationship satisfies $x = y$. Deviations from this equality indicate the presence of inequality. While this interpretation is intuitive and widely used, it relies on a restrictive assumption: the Lorenz criterion. Specifically, the Gini coefficient provides a consistent ordering of inequality between two distributions if and only if their Lorenz curves do not intersect. When Lorenz curves cross, the Gini-based ordering becomes ambiguous, limiting its applicability for comparing heterogeneity across such cases (Atkinson, 1970; James and Taeuber, 1985; Marshall, Olkin, and Arnold, 1979; Schwartz and Winship, 1980). Nunes, Trappenberg, and Alda (2020) discuss several other dispersion measures in the context of ecology, biology, and medicine. These measures exhibit other limitations which render them less practical the context of system heterogeneity.

To escape the Lorenz criterion, Weymark (1981) characterizes a class of generalized Gini absolute inequality indices:

$$Generalized.Gini(S_{\mathbb{P}}^{\theta}) = \mu(n) - \sum_{g=1}^{G} w_g \hat{n}_g,$$

with weights w_g satisfying $w_1 \leq w_2 \leq \cdots \leq w_g$, $\sum_{g=1}^{G} w_g = 1$, and $\hat{n}_g$ and $\mu(n)$ are as previously defined. Unlike the standard Gini coefficient, which yields meaningful comparisons only when Lorenz curves do not intersect, this generalized formulation induces total orders over systems with an equal number of groups, even in the presence of intersecting Lorenz curves. However, the very flexibility that enables such generality also limits the practical appeal of these indices, as the absence of a canonical weighting scheme complicates their interpretation and empirical application (Gajdos and Weymark, 2005; Weymark, 2003).

Other dispersion measures such as the Hoover index (Hoover, 1936) and the Atkinson index (Atkinson, 1970) focus on a normative approach to interpret the distance between the observed system and the uniform distribution. The Hoover index measures the average absolute distance between each household's income and the average household income, while the Atkinson index is a generalized measure that can specify the effect of changes to the distribution among the low-income households via a inequality-aversion parameter.

Cowell (2000) provides a comprehensive overview of inequality measures, including the discussion of desirable properties, technical features, and empirical applications. Lambert (2002) discusses the usage of inequality measures in taxation schemes and social welfare. A fundamental aspect of the discussion is the "inequality aversion assumed for the underlying social welfare function" (p. 6). And inequality aversion relies on a natural ordering of groups in a system. Such a normative approach is clearly suitable for inequality, but not necessarily practical in the case of heterogeneity where the groups in a system can be unordered.

2.3 Measures of Concentration

Concentration measures quantify the extent to which a system is dominated by its largest groups, thereby capturing the degree of dominance or aggregation within the distribution. In contrast, measures of heterogeneity are concerned with the overall dispersion or spread across all groups, rather than the relative size of the largest ones. Chakravarty and Eichhorn (1991) show that both the HHI and SE are derivatives of the *Hannah-Kay* class of concentration measures (Hannah and Kay, 1977), defined with the perception parameter α as:

$$H_{\alpha}^{G}\left(S_{\mathbb{P}}^{\theta}\right)=\begin{cases}\left[\sum\limits_{g=1}^{G} P_{g}^{\alpha}\right]^{\frac{1}{\alpha-1}} & \text{if } \alpha>0,\ \alpha\neq 1\\ \prod\limits_{g=1}^{G} P_{g}^{P_{g}} & \text{if } \alpha=1\end{cases}, \text{ where } P_{g}=\frac{n_{g}}{\sum\limits_{g=1}^{G} n_{G}}.$$

Consider a system $S_{\mathbb{P}}^{\theta}$ with $G=|\theta|$ groups. HHI (Herfindahl, 1950; Hirschman, 1945) and its complement, the Gini-Simpson index (GSI) (Gini, 1912; Simpson, 1949), of system S are defined as:

$$HHI\left(S_{\mathbb{P}}^{\theta}\right)=\sum_{g=1}^{G} P_{g}^{2}=H_{2}^{G}\left(S_{\mathbb{P}}^{\theta}\right), \quad GSI\left(S_{\mathbb{P}}^{\theta}\right)=1-HHI\left(S_{\mathbb{P}}^{\theta}\right).$$

SE is defined as:

$$SE\left(S_{\mathbb{P}}^{\theta}\right)=-\sum_{g=1}^{G}\left[P_{g}\cdot ln\left(P_{g}\right)\right]=-ln\left(H_{1}^{G}\left(S_{\mathbb{P}}^{\theta}\right)\right).$$

HHI induces a total order over systems and admits a cardinal interpretation: it represents the probability that two independent draws, with replacement, from the population will belong to the same group. Despite its intuitive appeal, HHI places disproportionate weight on changes in the sizes of larger groups. This property is advantageous in contexts where concentration is of primary interest, such as in the assessment of market power among firms (Chakravarty and Eichhorn, 1991; Herfindahl, 1950; Hirschman, 1945) or the influence of political parties (Laakso and Taagepera, 1979).

Consider, for example, the two systems (48,47,5) and (60,29,11). The GSI assigns values of 0.5462 and 0.5438 to these systems, respectively—suggesting a nearly identical degree of deconcentration.However, the underlying group structures differ in a substantively meaningful way: the first system comprises two dominant groups and a small residual group, while the second features a single dominant group with a more gradual decline in size among the remaining groups. his illustrates a case in which reliance on GSI may obscure important structural distinctions, limiting its informativeness as a measure of heterogeneity. In contrast, the DUH values for these systems are 0.2865 and 0.3170, respectively, indicating that the second system is substantially more heterogeneous. This example underscores the importance of restricting the use of HHI and GSI to contexts concerned with concentration rather than employing them as general-purpose measures of heterogeneity (Kvålseth, 2022). The simplicity of interpretation offered by HHI and GSI can, in such cases, mask salient differences between systems.

SE is another widely used measure of concentration, particularly in the information theory and rational inattention literature (Pomatto, Strack, and Tamuz, 2023; Sims, 2010, 2003). As a negative logarithmic transformation of $H_1^G\left(S_{\mathbb{P}}^{\theta}\right)$, SE differs in two important respects: (1) it lacks the cardinal interpretation of HHI; and (2) it exhibits different behavior under refinements of the group structure, particularly when existing groups are subdivided.

Moreover, both HHI/GSI and SE are invariant to the addition of zero-groups. While such invariance is intuitive in contexts involving market shares or uncertainty, it becomes problematic in contexts of assessing heterogeneity. Instead, GSI satisfies the so-called small-subgroups property, as shown by Chakravarty (2015). This property states that "the addition of a subgroup with population size smaller than that of the smallest of the existing subgroups, all other subgroups' population sizes held constant, will increase the value of the fractionalization index" (pp. 112–113). Notably, the proof requires that the added subgroups be non-empty, due to the zero-group invariance of the measure. DUH, on the other hand, satisfies the small-subgroups property and also decreases its value with the addition of zero-groups. This added sensitivity makes DUH well-suited for comparing systems with differing numbers of groups, and for capturing structural changes that are otherwise obscured by measures exhibiting zero-group invariance.

The precise ordering of heterogeneity across systems depends critically on the nature of the comparison and the degree of similarity between the systems under consideration. Contrary to the common idiom, comparisons between fundamentally different entities—such as apples and oranges—need not be inherently meaningless. When evaluated along well-defined and shared dimensions (e.g., density, color intensity, or sugar concentration), such comparisons can yield informative insights. It is even conceivable to assess both entities relative to a property characteristic of only one—for instance, citrus quality—though such an evaluation would predictably disadvantage the non-citrus item. However, certain comparisons remain intrinsically problematic. For example, comparing x to y when asserting that an apple is x times denser than an ice cube, and an orange is y times more round than a bowling ball.

To meaningfully compare heterogeneity across systems, it is essential to first establish the criteria that render such systems comparable. A key consideration is the inclusion of all potential group labels, including those associated with zero population. Incorporating zero-groups provides a consistent reference frame for evaluating heterogeneity. Accordingly, when comparing systems with differing numbers of groups, the system with fewer groups should be interpreted as implicitly containing additional groups with zero population. This approach ensures that heterogeneity is assessed over a common group structure, thereby enabling valid and interpretable comparisons.

Adapting a concentration measure to account for the inclusion of zero-groups is hardly revolutionary. Cracau and Lima (2016) show that a normalized version of HHI solves this issue by revising the formula to:

$$NHHI\left(S_{\mathbb{P}}^{\theta}\right) = \frac{HHI\left(S_{\mathbb{P}}^{\theta}\right) - \frac{1}{G}}{1 - \frac{1}{G}} \in [0, 1].$$

This measure improves system comparability by including the number of groups in its functional form at the of HHI's probabilistic interpretation. [1]

3 Axioms for Measures of Heterogeneity

As discussed above, the ordering of heterogeneity across systems can, in many cases, be determined flexibly to accommodate specific properties subject to some limitations. In this section, I propose a set of axioms that reflect desiderata more appropriate for the measurement of system heterogeneity. These axioms are intended to guide the construction and evaluation of heterogeneity measures that are sensitive to structural features beyond mere dispersion or concentration.

I begin by introducing the set fundamental axioms that induce a partial order over systems. These axioms establish conditions under which two systems should be regarded as equally heterogeneous, and how two systems should be ordered when one is derived from the other through an elementary transformation.

Axiom 1 (Group Symmetry (GSYM))**.** *A measure of heterogeneity* Φ *satisfies the property of Group Symmetry if for every permutation* $\pi\left(S^{\theta}_{\mathbb{P}}\right)$ *of* $S^{\theta}_{\mathbb{P}}$, $\Phi\left(S^{\theta}_{\mathbb{P}}\right)=\Phi\left(\pi\left(S^{\theta}_{\mathbb{P}}\right)\right)$.

GSYM requires that the heterogeneity of a system in invariant to a reassignment of group labels, holding the original partitions fixed. For example, take $n_a, n_b, n_c \in \mathbb{N}$,

θ_g \ $S^{\theta}_{\mathbb{P}}$	S^{θ}_{a}		S^{θ}_{b}		S^{θ}_{c}
θ_1	n_1		n_2		n_3
θ_2	n_2	$\sim$	n_1	$\sim$	n_2
θ_3	n_3		n_3		n_1

GSYM ensures that a measure of heterogeneity depends solely on the distribution of group sizes, not on the specific labels assigned to those groups. If two systems have the same number of groups and their group size distributions are permutations of each another, they should be assigned the same heterogeneity value. Satisfying GSYM allows for meaningful comparisons between systems defined over different group label sets, provided the cardinality of the set of group labels is identical. This axiom thus abstracts away from group identity and focuses exclusively on structural features of the distribution.

Axiom 2 (Scale Invariance (INV))**.** *A measure of heterogeneity* Φ *satisfies the property of Scale Invariance if for any system* $S^{\theta}_{\mathbb{P}}$ *and a scalar* $\lambda \in \mathbb{R}_{++}$, $\Phi(S^{\theta}_{\mathbb{P}})=\Phi\left(\lambda\cdot S^{\theta}_{\mathbb{P}}\right)$.

[1]Consider the following two systems:

$$S^{\theta}_{1}=(0.4,0.4,0.2) \text{ and } S^{\theta}_{2}=(0.5,0.3,0.1,0.1).$$

These two systems have the same HHI (0.36), but they have different NHHIs ($NHHI(S,G=3)=0.04$ and $NHHI(S',G=4)\approx 0.15$). By observation, it may not be clear whether S_1 and S_2 are equally homogeneous, but a comparison of these two NHHIs is unlikely to be convincing. Once we account for zero-groups and make $S_1=(0.4,0.4,0.2,0)$, the NHHIs of the two systems are the same (0.15)—just like their HHIs—but the level of heterogeneity can no longer be intuitively interpreted.

Scale Invariance (INV) permits the generalization of systems from vectors of non-negative integers to vectors of non-negative real numbers. While systems are initially defined as partitions of finite populations—represented by integer-valued group sizes—satisfaction of INV allows the heterogeneity measure to remain well-defined under continuous scaling. This axiom ensures that heterogeneity depends solely on the relative distribution of group sizes, not on their absolute magnitudes. Consequently, the measure is invariant to unit conversions (e.g., from individuals to percentages), and its domain naturally expands from partitions of finite sets to partitions over uncountably infinite sets. Under INV, any system can be represented as a vector of non-negative real numbers summing to unity, facilitating broader applicability and interpretability.

Axiom 3 (Principle of Transfers (PT)). *Let $\hat{S}^{\theta}_{\mathbb{P}}$ be a permutation of $S^{\theta}_{\mathbb{P}}$ such that $\hat{n}_1 \geq \hat{n}_2 \geq \cdots \geq \hat{n}_G$ and let e^G_i be a length G vector where only the ith element is 1 and the rest are 0. A measure of heterogeneity Φ satisfies the Principle of Transfers if $i > j, \hat{n}_i - 1 \geq \hat{n}_j + 1, \hat{n}_i - 1 \geq \hat{n}_{i+1}, \hat{n}_{j-1} \geq \hat{n}_j + 1$ implies $\Phi\left(\hat{S}^{\theta}_{\mathbb{P}}\right) < \Phi\left(\hat{S}^{\theta}_{\mathbb{P}} - e_i + e_k\right)$.*

PT requires that, holding the order of groups constant, a transfer from a larger group to a smaller group increases heterogeneity. For example, take $n_1, n_2, n_3 \in \mathbb{N}$ such that $n_1 \geq n_2 \geq n_3$ and $\varepsilon < \min\left\{n_2 - n_3, \dfrac{n_1 - n_2}{2}\right\}$.

θ_g \ $S^{\theta}_{\mathbb{P}}$	S^{θ}_a		S^{θ}_b		S^{θ}_c
θ_1	n_1		$n_1 - \varepsilon$		$n_1 - \varepsilon$
θ_2	n_2	$\prec$	$n_2 + \varepsilon$	$\prec$	n_2
θ_3	n_3		n_3		$n_3 + \varepsilon$

Here, S^{θ}_b is obtained from S^{θ}_a by transferring ε from the largest group to the second largest, and S^{θ}_c is obtained by transferring ε from the largest group to the smallest. In both cases, PT requires that heterogeneity increases: $S^{\theta}_a \prec S^{\theta}_b \prec S^{\theta}_c$.

This axiom was first formalized by Dalton (1920) in the context of income inequality, stating: "If there are only two income receivers, and a transfer occurs from the richer to the poorer, inequality is reduced" (p. 351). It is commonly referred to as the Pigou–Dalton transfer principle, and has long been regarded as a foundational requirement for inequality measures (Dalton, 1920; Pigou, 1912; Weymark, 1981).

Kolm (1976) shows that any heterogeneity measure Φ satisfying GSYM, INV, and PT induces a total ordering over systems that satisfy the Lorenz criterion. That is, all measures discussed in the previous section agree in their ordering of systems whose Lorenz curves do not intersect. However, when the domain is extended to include systems whose Lorenz curves intersect, the axioms of GSYM, INV, and PT are no longer sufficient to induce a total order. In such cases, additional axioms, reflecting more refined normative judgments, are required to fully characterize the heterogeneity ordering across systems.

3.1 The Two Determinants of Heterogeneity

The Gini coefficient, HHI, and SE satisfy GSYM because their functional forms treat all groups identically. However, GSYM can be satisfied under a more nuanced interpretation: by treating each *type* of groups, rather than each group, identically. In any system, there is always a largest group and a set of remaining groups. A measure that distinguishes the largest group from the rest—while treating all groups of the same type symmetrically—can still satisfy GSYM.

This perspective introduces a useful decomposition of heterogeneity into two distinct determinants: (1) the dominance of the largest group, and (2) the distribution among the remaining groups. If a heterogeneity measure is constructed such that the influence of the largest group is orthogonal to that of the rest, then changes in the overall measure can be interpreted with equivalent changes in either component while holding the other constant. This decomposition enhances interpretability and allows for more granular analysis of structural variation across systems.

For clarity and convenience in the discussion that follows, I refer to the group with the largest population share in a system as the largest group, and all other groups as minority groups. To streamline subsequent definitions and formalizations, I introduce the following notational conventions.

Definition 8 (Ordered system $\hat{S}^{\theta}_{\mathbb{P}}$)**.** *Let $S^{\theta}_{\mathbb{P}}$ be an arbitrary system. An ordered system $\hat{S}^{\theta}_{\mathbb{P}} = (\hat{n}_1, \ldots, \hat{n}_G)$ is the permutation of $S^{\theta}_{\mathbb{P}}$ such that $\hat{n}_1 \geq \hat{n}_2 \geq \cdots \geq \hat{n}_G$.*

Definition 9 ($\hat{P}_g$, $\tilde{P}_g$)**.** *For any ordered system $\hat{S}^{\theta}_{\mathbb{P}}$ with G groups, define:*

$$\hat{P}_g = \frac{\hat{n}_g}{\sum\limits_{g=1}^{G} \hat{n}_g} \quad \textit{and} \quad \tilde{P}_g = \frac{\hat{n}_g}{\sum\limits_{g=2}^{G} \hat{n}_g}.$$

$\hat{P}_g$ is the relative size of group θ_g in the population, and $\tilde{P}_g$ is the relative size of group θ_g in the population that excludes the largest group.

Under this framework, a measure of heterogeneity Φ satisfying GSYM can be expressed as a composite function $\Phi = \Phi(\phi, \psi)$, where $\phi\left(\hat{S}^{\theta}_{\mathbb{P}}\right)$ captures the contribution of the largest group, and $\psi\left(\hat{S}^{\theta}_{\mathbb{P}}\right)$ captures the contribution of the minority groups to the overall heterogeneity of the system. This decomposition enables the formulation of characterization axioms that separately address the structural roles of the largest group and the minority groups, thereby introducing greater flexibility in the design of heterogeneity measures.

The characterization axioms introduced below, combined with the fundamental axioms, induce a total order over all systems and uniquely characterize the family of measures DUH. As with existing axiomatizations of measures such as the HHI and SE, my approach emphasizes cardinal interpretations and specifies how the measures respond to structural changes. However, the type-of-group framework allows for a more nuanced treatment of group structure. Specifically, it distinguishes between the contributions of the largest group and the minority groups, enabling the design of axioms that reflect the dual determinants of heterogeneity.

Axiom 4 (Independence (IND))**.** *A measure of heterogeneity Φ satisfies Independence if it is a composite function $\Phi(\phi, \psi)$ where $\phi\left(\hat{S}^{\theta}_{\mathbb{P}}\right) = \phi\left(\hat{P}_1\right)$ and $\psi\left(\hat{S}^{\theta}_{\mathbb{P}}\right) = \psi\left(\hat{P}_2, \ldots, \hat{P}_G\right)$*

IND requires that the contributions of the largest group and the minority groups to overall heterogeneity be treated independently. In particular, it allows for the interpretation of any changes in heterogeneity to equivalent changes resulting from adjustments solely to either the size of the largest group or the distribution among the minority groups. The applicability of IND to systems with any number of groups necessitates a specific classification between the largest group and all remaining groups.[2]

Definition 10. *Take $p \in [1, \infty)$, $c_1 \in \mathbb{R}$, and $c_2 \in \mathbb{R}_{++}$. The function $\psi_p : \mathbb{R}_+^{G-1} \to \mathbb{R}$ defined as:*

$$\psi_p(S) = \psi\left(\tilde{P}_2, \ldots, \tilde{P}_G\right) = 1 - \left(\sum_{g=2}^{G}\left|\tilde{P}_g - \frac{1}{G-1}\right|^p\right)^{\frac{1}{p}},$$

is a measure of evenness in the distribution of minority groups.

To capture features of the minority groups' distribution with a function ψ_p, I define it to be a function of the L^p-norm between the observed distribution in the minority groups and the uniform distribution in the minority groups. The L^p-norms with $p \in [1, \infty)$ and the set of systems with G groups form a well-defined metric space. Since $\tilde{P}_g \in [0, 1]$ and $\sum_{g=2}^{G} \tilde{P}_g = 1$ by construction, the L^p-norm between the minority distribution and the uniform distribution is always between zero and unity. The specific affine transformation on the L^p-norm makes it such that $\psi_p\left(S_{\mathbb{P}}^{\theta}\right) = 0$ corresponds to a system with a minimally heterogeneous minority distribution and $\psi_p\left(S_{\mathbb{P}}^{\theta}\right) = 1$ corresponds to a system with a maximally heterogeneous minority distribution.

In pursuit of some cardinal interpretation of changes in Φ when ψ_p is fixed, I introduce a minimalist refinement of PT that builds on IND: the principle of proportional transfers.

Axiom 5 (Principle of Proportional Transfers (PPT))**.** *Let $\hat{S}_{\mathbb{P}}^{\theta}$ be an arbitrary ordered system of G groups. Let e_i^G be a length G vector where only the ith element is 1 and the rest are 0. A measure of heterogeneity Φ of $\hat{S}_{\mathbb{P}}^{\theta}$ satisfies the Principle of Proportional Transfers if $c \in \mathbb{R}_+,\ \alpha \in \mathbb{R}_{++}$*

$$\frac{\hat{n}_1 - c}{\hat{n}_1 + \hat{n}_2 + \cdots + \hat{n}_G} = \left(\frac{\hat{n}_1}{\hat{n}_1 + \hat{n}_2 + \cdots + \hat{n}_G}\right)^{\alpha} \text{ and } \hat{n}_1 - c \geq \hat{n}_2 + \tilde{P}_2 \cdot c$$

$$\Rightarrow \Phi\left(\hat{S}_{\mathbb{P}}^{\theta} - c \cdot e_1^G + \sum_{g=2}^{G} \tilde{P}_g c \cdot e_g^G\right) = \alpha \cdot \Phi\left(\hat{S}_{\mathbb{P}}^{\theta}\right)$$

PPT specifies that, holding the order of groups constant, a proportional transfer from the largest group to the minority groups—such that the size of the largest group is reduced from $\hat{P}_1$ to $\left(\hat{P}_1\right)^{\alpha}$, for some $\alpha \in \mathbb{R}_{++}$—should increase heterogeneity by a factor of α. In other words, any measure satisfying PPT responds to reductions in the size of the largest group in a tractable and predictable manner, provided that (1) the distribution among the minority groups remains unchanged, and (2) the original largest group retains its status as the largest group after the transfer. This axiom formalizes the intuition that diminishing dominance, while preserving internal minority structure, increases heterogeneity in an interpretable way. For example, when comparing the heterogeneity of S_a^{θ} and S_b^{θ} such that

[2] Alternative categorizations—such as separating the two largest groups from the rest—would be problematic when there are only two or three groups.

$$
\begin{array}{|c|}\hline \diagdown\; S_{\mathbb{P}}^{\theta} \\ \theta_g \\ \hline \theta_1 \\ \hline \theta_2 \\ \hline \theta_3 \\ \hline \end{array}
\quad
\begin{array}{|c|}\hline S_a^{\theta} \\ \hline \hat{P}_1 \\ \hline \hat{P}_2 \\ \hline \hat{P}_3 \\ \hline \end{array}
\prec
\begin{array}{|c|}\hline S_b^{\theta} \\ \hline \hat{P}_1^{\alpha} \\ \hline \hat{P}_2 + \frac{\hat{P}_2}{\hat{P}_2+\hat{P}_3}\left(\hat{P}_1 - \hat{P}_1^{\alpha}\right) \\ \hline \hat{P}_3 + \frac{\hat{P}_3}{\hat{P}_2+\hat{P}_3}\left(\hat{P}_1 - \hat{P}_1^{\alpha}\right) \\ \hline \end{array}
$$

, PPT prescribes that

$$\Phi\left(S_a^{\theta}\right) < \alpha\Phi\left(S_a^{\theta}\right) = \Phi\left(S_b^{\theta}\right).$$

PPT endows changes in the heterogeneity measure a cardinal interpretation. Specifically, the relationship $2\Phi\left(S_a^{\theta}\right) = \Phi\left(S_b^{\theta}\right)$ can be interpreted as "S_b^{θ} is twice as heterogeneous as S_a^{θ}." This interpretation holds because S_b^{θ} has the same value of heterogeneity as would result from reducing the largest group proportion in S_a^{θ} from $\hat{P}_1$ to $\left(\hat{P}_1\right)^2$, while maintaining both the evenness of the minority group distribution and the dominance of the original largest group. Thus, PPT provides a tractable and interpretable link between proportional reductions in group dominance and proportional increases in measured heterogeneity.

Axiom 6 (Contractibility (CON))**.** *Take an arbitrary ordered system $\hat{S}_{\mathbb{P}}^{\theta}$ of G groups. Denote $\hat{S}_{\mathbb{P}}^{\theta\cup\{\theta_{G+1}\}} = (\hat{n}_1, \hat{n}_2, \ldots, \hat{n}_G, 0)$. A measure of heterogeneity Φ satisfies Contractibility if*

$$\hat{n}_2 > 0 \;\Rightarrow\; \Phi\left(\hat{S}_{\mathbb{P}}^{\theta\cup\{\theta_{G+1}\}}\right) < \Phi\left(\hat{S}_{\mathbb{P}}^{\theta}\right)$$

CON requires that, upon extending a system by introducing a new group label such that the resulting partition is a zero-group, the overall heterogeneity of the system must decrease. A key implication of this axiom is that any comparison of heterogeneity across systems implicitly assumes a common group structure, even if some groups are unpopulated. For example, the distribution $\left(\frac{1}{2}, \frac{1}{2}\right)$ is maximally heterogeneous among two-group systems, while $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is maximally heterogeneous among three-group systems. By CON, comparing these two requires interpreting the former as $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ which is less heterogeneous than $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Axiom 7 (Unity (UNI))**.** *Take an arbitrary ordered system $\hat{S}_{\mathbb{P}}^{\theta}$ of G groups. A measure of heterogeneity Φ satisfies Unity if*

$$\Phi\left(S_{\mathbb{P}}^{\theta}\right) = 0 \iff \hat{P}_1 = 1$$

and

$$\Phi\left(S_{\mathbb{P}}^{\theta}\right) = 1 \iff \hat{P}_1 = \hat{P}_2 = \cdots = \hat{P}_G = \frac{1}{G}$$

The axiom UNI serves as a normalization condition, requiring that the heterogeneity measure $\Phi\left(S_{\mathbb{P}}^{\theta}\right)$ attains the value 0 for a minimally heterogeneous system and 1 for a maximally heterogeneous system. A measure satisfying UNI thus enables meaningful baseline comparisons, allowing the heterogeneity of any given system to be evaluated relative to these two reference points.

The Gini coefficient, the HHI, and SE satisfy all of the fundamental axioms as well as some of the characterization axioms. The additional axioms used to characterize HHI and SE are listed in Table 1 and formally defined in Appendix C.

4 Characterizing The Descriptive Units of Heterogeneity

Proposition 1. *Let $S^{\theta}_{\mathbb{P}}$ be an arbitrary system with more than three groups. Consider a measure of heterogeneity $\Phi_p\left(\hat{S}^{\theta}_{\mathbb{P}}\right) = \Phi\left(\phi\left(\hat{S}^{\theta}_{\mathbb{P}}\right), \psi_p\left(\hat{S}^{\theta}_{\mathbb{P}}\right)\right)$ that satisfies GSYM, INV, and IND. Holding $\hat{P}_1$ constant, if Φ is strictly increasing in ψ_p, then $\Phi\left(\phi\left(\hat{P}_1\right), \psi_p\right)$ satisfies PT if and only if $p > 1$.*[3]

Definition 11. *Let $\hat{S}^{\theta}_{\mathbb{P}} = (\hat{n}_1, \hat{n}_2, \ldots, \hat{n}_G)$ be a permutation of an arbitrary system $S^{\theta}_{\mathbb{P}}$ such that $\hat{n}_1 \geq \hat{n}_2 \geq \cdots \geq \hat{n}_G$. Denote $\hat{P}_1 = \frac{\hat{n}_1}{\hat{n}_1+\hat{n}_2+\cdots+\hat{n}_G}$ and $\tilde{P}_g = \frac{\hat{n}_g}{\hat{n}_2+\cdots+\hat{n}_G}$ where $g \in \{2, \ldots, G\}$ and $p > 1$. The family of descriptive units of heterogeneity (DUH) of the system $S^{\theta}_{\mathbb{P}}$ with $G \geq 2$ groups is:*

$$DUH\left(S^{\theta}_{\mathbb{P}}\right) = \frac{ln\left(\hat{P}_1\right)}{ln(G)} \cdot \left[\left(\sum_{g=2}^{G}\left|\tilde{P}_g - \frac{1}{G-1}\right|^p\right)^{\frac{1}{p}} - 1\right].$$

Theorem: *The descriptive units of heterogeneity constitute a uniquely determined family of measures—up to positive scalar multiples—that incorporate evenness through the function ψ_p, and satisfy the axioms of group symmetry, scale invariance, independence, the principle of transfers, the principle of proportional transfers, and contractibility.*

Proof sketch[4]

Step 1: I show that any index using an ordered system divided by the population size as inputs satisfies GSYM and INV.

Step 2: I then show that any $\Phi(\phi, \psi_p)$ that satisfies GSYM, INV, and PPT must be a positive monotonic transformation of $\frac{1}{\hat{P}_1}$.

Step 3: I show tha if an $\Phi(\phi, \psi_p)$ satisfies INV, IND, and PPT, then Φ must be multiplicatively separable. In other words, $\Phi = \phi \cdot \psi_p$.

Step 4: I show that, holding $\hat{P}_1$ constant and assuming GSYM, INV, and IND, the measure Φ, using the measure of evenness ψ_p as defined, satisfies PT if and only if $p > 1$ when $G > 3$ and $p \geq 1$ when $G = 3$.

Step 5: I show that if $\Phi(\phi, \psi_p)$ satisfies GSYM, INV, IND, and PPT, then $\phi = -cln\left(\hat{P}_1\right)$, $c \in \mathbb{R}_{++}$.

Step 6: For the case of $p = 2$, using the Euclidean distance for ψ_p, I show that the family DUH satisfies PT, in addition to GSYM, INV, IND, and PPT, by taking the derivative of the extreme case in which $\hat{P}_1$ is close to 1 and $\psi = 1$ with respect to a transfer from the largest group to the second-largest group.

Step 7: I show that the family DUH satisfies CON and UNI.

[3]This proposition suggests that we need to be careful regarding the functional form of $\phi\left(\hat{P}_1\right)$. To satisfy PT, it must be that when there is a transfer from the largest group to a minority group, the increase in $\phi\left(\hat{P}_1\right)$ must dominate the decrease in ψ_p in the case where evenness decreases. Also, if $G = 3$, this proposition holds with $p \geq 1$.

[4]Full proof in Appendix B.

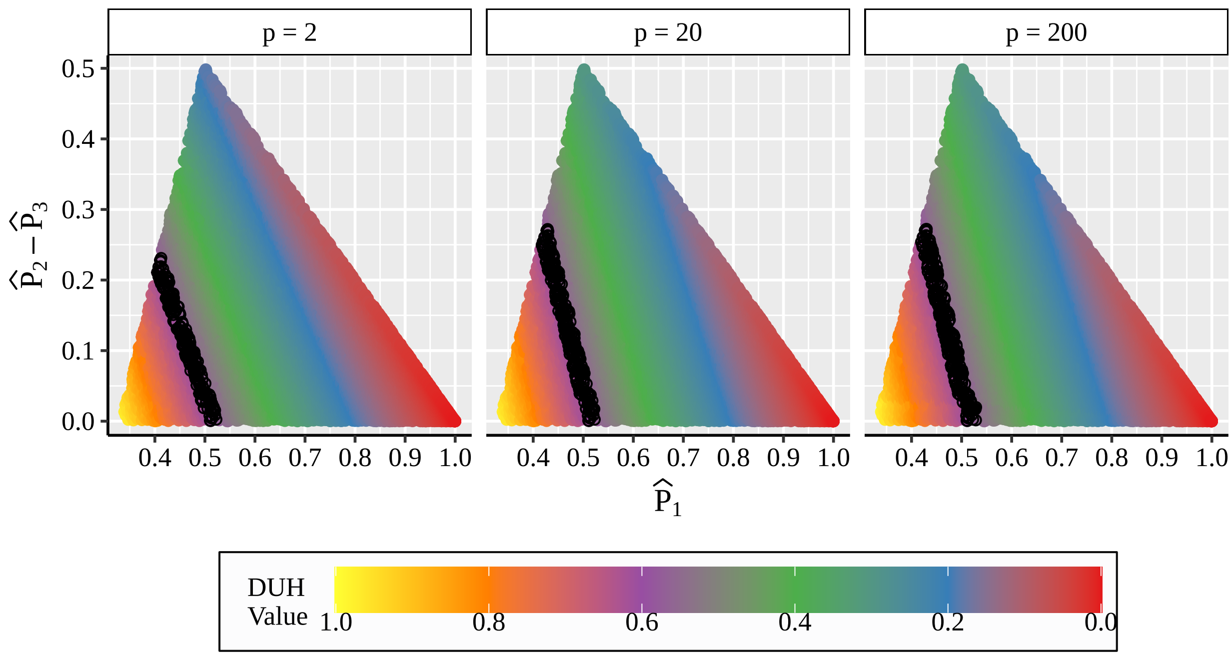
4.1 The Role of The Evenness Parameter p

Notice that the evenness parameter p, from the function ψ_p, determines the contribution of evenness in DUH. As p increases, minority groups that are farther away from $\frac{1}{G-1}$ are weighted more. As $p \to \infty$,

$$L^p \to L^\infty = \max_{g \in \{1,\ldots,G\}} \left\{ \left| P_g - \frac{1}{G-1} \right| \right\}.$$

In words, for sufficiently large values of the evenness parameter p, the function ψ_p becomes increasingly sensitive to the most extreme deviations from evenness, such that the heterogeneity measure is effectively determined by either the largest or the smallest minority group. As illustrated in Figure 1, the variation in DUH values, conditional on a fixed value of $\hat{P}_1$, decreases with increasing p. This reflects the fact that, as the power parameter in the L^p-norm increases, ψ_p assigns greater weight to the deviation of the most extreme group share from the uniform benchmark $\frac{1}{G-1}$. The systems highlighted by the black band correspond to those with DUH values in the interval ([0.58, 0.62]. As p increases, the slope of this band becomes steeper, indicating a reduction in the range of DUH values associated with a given $\hat{P}_1$.

Figure 1: DUH with Evenness Parameter Set to 2, 20, and 200



4.2 Comparing DUH to Other Measures

Table 1 shows which axioms are satisfied by the Gini coefficient, DUH, HHI, and SE. Although conceptually distinct from the Gini coefficient, DUH satisfy all axioms that the Gini coefficient does and more; the additional characterization axioms enable DUH to induce a total order over arbitrary systems, thereby overcoming the limitations imposed by the Lorenz criterion.

In contrast, the differences in axiomatic properties among DUH, HHI, and SE reflect the distinct normative foundations and intended applications of each measure. Specifically, while HHI and SE satisfy expandability, i.e., invariance to the inclusion of zero-population groups, DUH do not. Both HHI and SE satisfy their own additional characterization axioms, endowing these measures unique cardinal interpretations of concentration.[5]

Importantly, all measures considered here can be affine-transformed, for a fixed $|\theta|$, to satisfy the normalization axiom UNI. As a measure of heterogeneity, DUH complements inequality and concentration measures in applied contexts by fulfilling distinct analytical objectives.

Table 1: Measures and Axioms

Type	Axiom	Gini	DUH	HHI	SE
Fundamental	Group symmetry (GSYM)	✓	✓	✓	✓
	Scale invariance (INV)	✓	✓	✓	✓
	Principle of transfers (PT)	✓	✓	✓	✓
Characterization[4]	Independence (IND)	×	✓	✓	✓
	Principle of proportional transfers (PPT)	×	✓	×	×
	Contractibility (CON)	✓	✓	×	×
	Unity (UNI)	✓	✓	×	×
	Expandability	×	×	✓	✓
	Replication principle	×	×	✓	×
	Shannon's additivity	×	×	×	✓

When the Lorenz curves of two arbitrary systems do not intersect, the Gini coefficient, DUH, GSI, and SE induce the same order over those systems. Consequently, the rank correlations between these measures, and more generally among all Schur-convex measures, are expected to be high. Using the permutation of distinct ordered systems with 100 total elements, Table 2 presents the correlation and rank correlation matrices for the case where $|\theta| = 3$, an arbitrary choice for the number of groups.[6]

Table 2: Correlation Matrices with $|\theta| = 3$, $|\mathbb{P}| = 100$

Correlation Matrix

	DUH	Gini	GSI	SE
DUH	1	0.957	0.832	0.874
Gini		1	0.947	0.953
GSI			1	0.984
SE				1

Rank Correlation Matrix

DUH	Gini	GSI	SE
1	0.974	0.957	0.987
	1	0.9951	0.987
		1	0.985
			1

However, the key contribution of DUH lies in its ability to differentiate between systems that do not satisfy the Lorenz criterion. Figure 2 illustrates this distinction by plotting the rank percentiles of the Gini coefficient, GSI, and SE against those of DUH for systems with three groups and a total

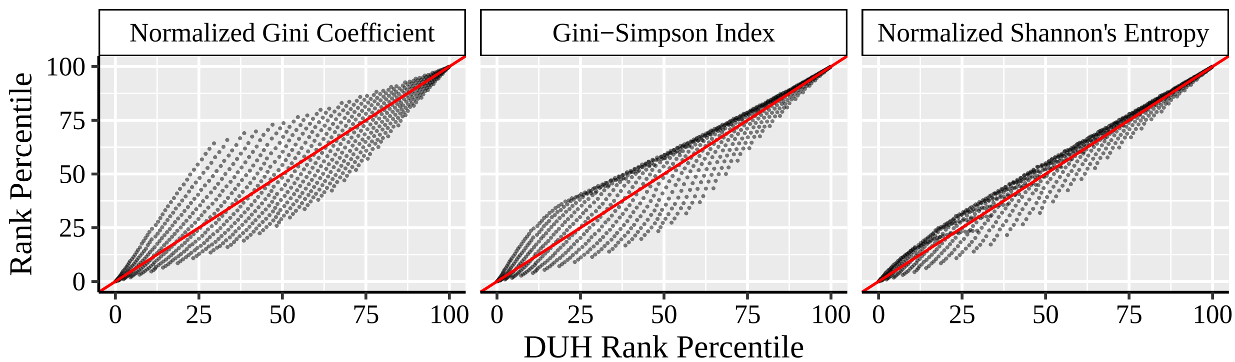
[5] See Appendix B for the definitions of Expandability, the Replication Principle, and Shannon's Additivity, as well as the unique characterizations these axioms provide.

[6] One can verify that similar correlation patterns hold for other values of $|\theta|$. See Table A1 for the case with ten groups.

of 100 elements.[7]

In the axes of Figure 2, a higher rank percentile means the measure evaluates a system to be more heterogeneous. The red diagonal lines indicate systems that receive identical absolute rank percentiles under DUH and the comparison measure. Any pair of points connected by a line with negative slope corresponds to two systems whose heterogeneity rankings are reversed between DUH and the stated measure. According to the Lorenz criterion, a transfer from a larger group to a smaller group increases heterogeneity; thus, as the size of the largest group decreases, the number of distinct systems that are more heterogeneous necessarily declines. Consequently, discrepancies in rankings between DUH and the other measures are more prevalent at the lower end of the heterogeneity spectrum than at the upper end.

Figure 2: Rank Correlation between Measures for $|\theta| = 3$, $|\mathbb{P}| = 100$



To elucidate the source of relative rank differences, Figure 3 presents a direct comparison between DUH and each of the Gini coefficient, GSI, and SE. For consistency across panels, the Gini coefficient, GSI, and SE are affine-transformed to satisfy UNI.

The horizontal axis in Figure 3 represents the size of the largest group in the system, denoted $\hat{P}_1$, while the vertical axis captures the difference between the second-largest and smallest group sizes, $\hat{P}_2 - \hat{P}_3$. For any fixed value of $\hat{P}_1$, the GSI exhibits limited variation along the vertical axis, indicating insensitivity to differences between minority group sizes. In contrast, the variation in SE depends on the value of $\hat{P}_1$; when $\hat{P}_1$ is sufficiently close to $\frac{1}{3}$ or 1, SE becomes nearly invariant. DUH, by comparison, display substantial variation along both axes and is approximately constant only when $\hat{P}_1$ and $\hat{P}_2 - \hat{P}_3$ move in opposite directions.

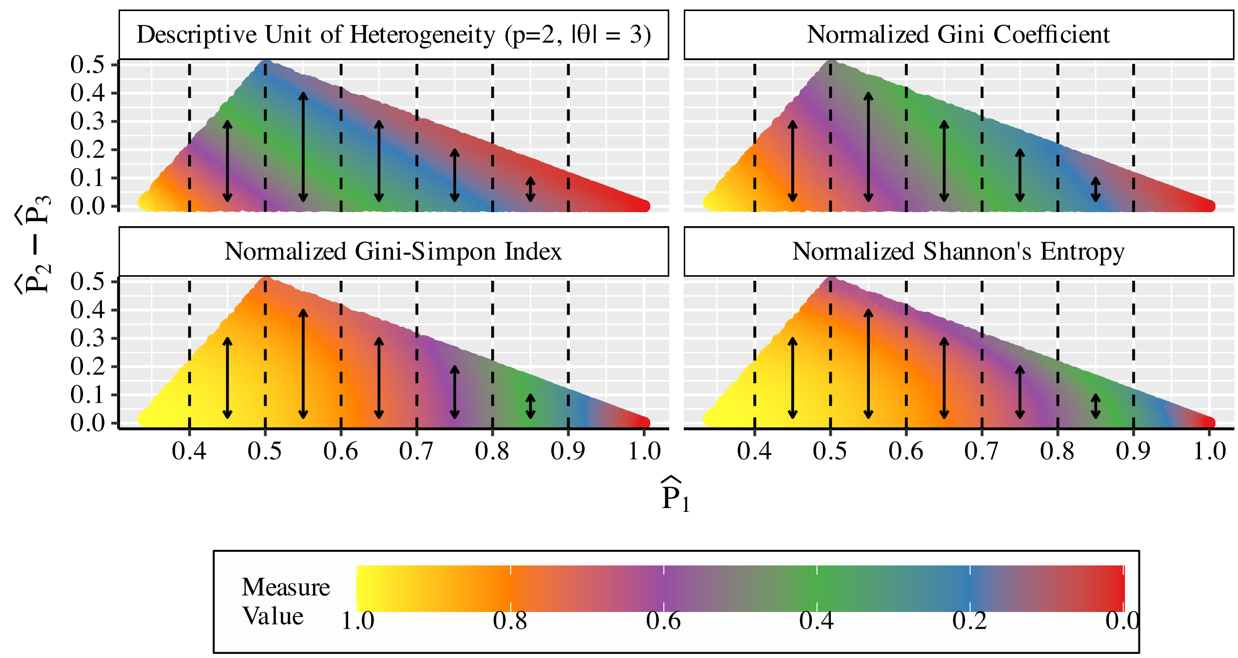
5 Empirical Applications of DUH

DUH, as a family of parsimonious descriptions of heterogeneity in systems with discrete distributions over unordered population groups, lend themselves to a wide range of empirical applications. This section presents two illustrative examples that demonstrate the practical utility of DUH in applied settings. As in the previous section, the Gini coefficient, GSI, and SE are normalized to lie within the unit interval $[0, 1]$, wherever applicable, to facilitate direct comparison across measures.

Before proceeding, it is important to emphasize the role of researcher discretion in defining the set of group labels θ. The empirical relevance and interpretability of DUH—and indeed of

[7]The analogous figure for systems with ten groups is provided in Figure B1.

Figure 3: Comparison between DUH, Gini, GSI, and SE with $|\theta| = 3$ and $p = 2$



any heterogeneity measure—depend critically on the construction of this group partition. In all applications that follow, the DUH measure employed uses the evenness parameter $p = 2$.

5.1 Reasonable Collection of Groups θ

Satisfaction of the CON axiom requires that the set of group labels, θ, be constructed with care, as each element $\theta_g \in \theta$ must be meaningfully comparable to the others. Figure 4 illustrates this principle using a practical example: the evolution of the racial composition of the San Francisco Metropolitan Statistical Area from 2017 to 2022, based on the American Community Survey (ACS) 1-year data (Ruggles et al., 2024).

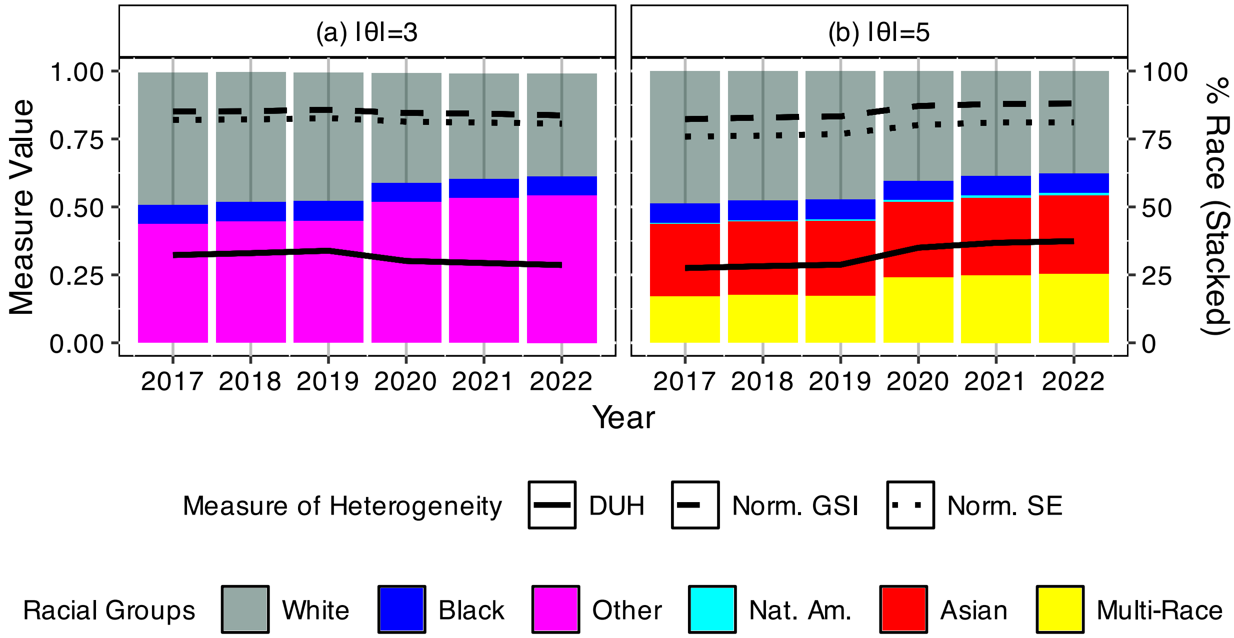
In panel (a), the set of group labels is $\theta = \{\text{White}, \text{Black}, \text{Other}\}$, aggregating all non-White, non-Black individuals into a single residual category. In panel (b), the "Other" category is disaggregated into three distinct subgroups, yielding $\theta = \{\text{White}, \text{Black}, \text{Asian}, \text{Native American}, \text{Multi-Race}\}$.

In panel (a), the three-group heterogeneity of San Francisco remains relatively stable over time, driven by a decline in the White population and a corresponding increase in the "Other" population. Heterogeneity begins to decline after 2019, when the "Other" group surpasses the White population as the largest group and continues to grow. This shift highlights the importance of the GSYM axiom, which enables researchers to interpret heterogeneity as a distributional property independent of group labels.

However, the inference changes once the group label θ is redefined to capture finer distinctions among subgroups. In panel (b), where the Asian, Native American, and Multi-Race populations are considered separately, heterogeneity increases after 2019, reflecting the proportional growth of these subgroups while shrinking the White population, holding the order of group sizes constant. Such distributional changes are precisely what the PPT axiom is designed to capture.

As this example illustrates, there is no universally correct specification of θ. The determination

Figure 4: Different Collection of Groups Can Yield Opposite Inferences



of group labels is a framing decision left to the discretion of the researcher. Just as the use of the Gini coefficient presupposes the Lorenz criterion, the application of any heterogeneity measure requires a justified and contextually appropriate choice of group labels. In the case shown in Figure 4, a simple refinement of a residual category alters the inference, underscoring the importance of careful consideration in defining a reasonable θ.

With these considerations in place, we now turn to examples that illustrate when and why DUH measures are useful. The following applications show how DUH captures distributional features that other measures may overlook, particularly in settings where proportional differences matter.

5.2 Using DUH for Racial Heterogeneity

Consider a hypothetical city composed of three racial groups: White, Black, and Other. Table 3 reports the population shares and corresponding heterogeneity measures across five decades. Over time, the White population steadily increases, while the distribution between the two minority groups becomes more balanced.

Under this configuration, DUH increases monotonically, reflecting the growing evenness between the minority groups. In contrast, GSI decreases due to the rising dominance of the White population. SE initially increases but declines in the final period. These divergent trends underscore the distinct properties of each measure.

The key limitation of GSI and SE is that their sensitivity to elementary transfers is highly positively correlated with group size. In this example, DUH mitigates that bias by capturing changes in the distribution among minority groups, even when those groups are relatively small. This highlights DUH's utility in contexts where minority group dynamics are of substantive interest.

[8]All measure values are rounded to second digit after the decimal.

[9]Note that all systems here have intersecting Lorenz criterion so the Gini coefficient cannot be used for inference.

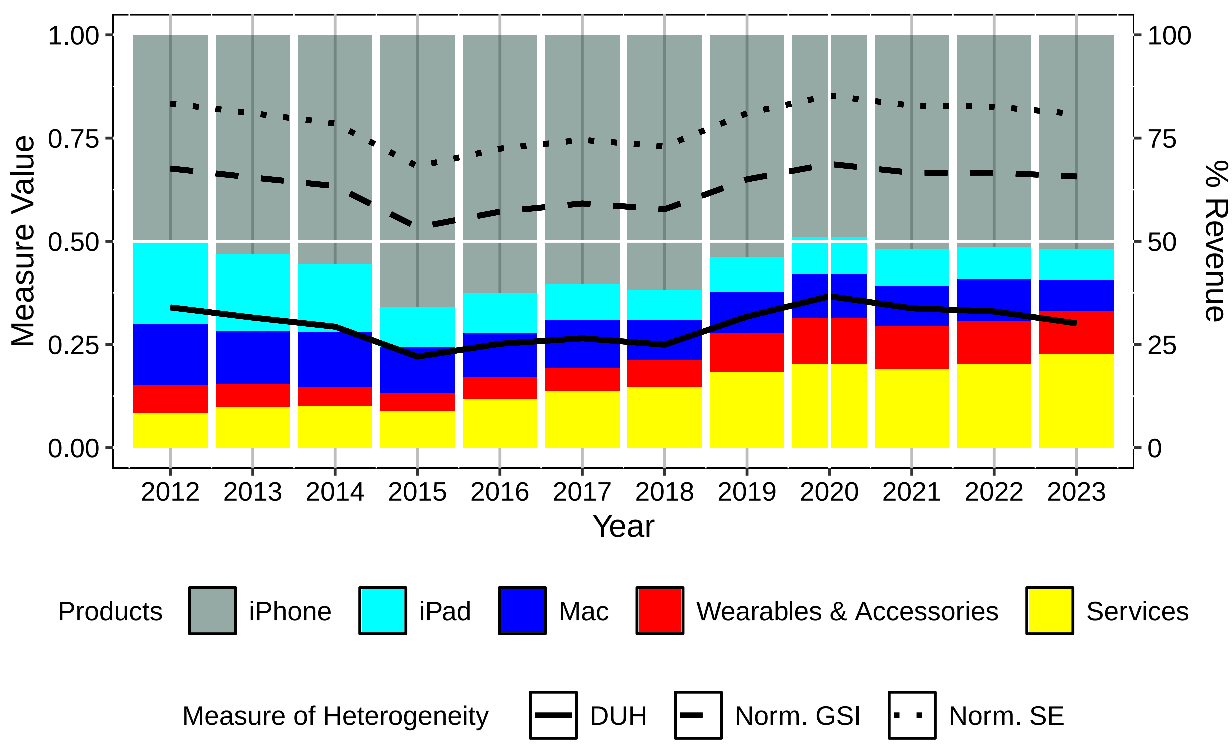
Table 3: The Progression of Racial Composition and Racial Heterogeneity of a Hypothetical City

Share (%)	Decade				
	1	2	3	4	5
White	55%	60%	65%	66%	69%
Black	42%	34%	26%	22%	17%
Other	3%	6%	9%	11%	14%
DUH[8]	0.21	0.23	0.26	0.28	0.31
Gini[9]	0.48	0.46	0.45	0.45	0.44
Norm. GSI	0.78	0.78	0.75	0.75	0.71
Norm. SE	0.73	0.77	0.77	0.78	0.76

5.3 Using DUH for Revenue Heterogeneity

DUH can also be applied to measure the extent of diversification in a firm's revenue portfolio. Consider Apple Inc., whose revenue is generated from five major product categories: iPhone, iPad, Mac, Wearables & Accessories, and Services. Figure 5 compares DUH with GSI and SE in this context, where concentration measures are commonly used to assess revenue diversification (Apple and Statista, 2024).

Figure 5: Revenue Heterogeneity using DUH, GSI, and SE



While the three measures generally move in tandem, a notable divergence occurs between 2020 and 2023. During this period, the revenue shares of Services and Wearables & Accessories increased without a corresponding decline in the share of iPhone revenue. This shift reduced the evenness among the minority product categories. DUH captures this change through a continuous and substantial decline, whereas the decrease in GSI is comparatively limited. This example highlights DUH's sensitivity to internal distributional changes, even when the dominant category remains stable.

6 Conclusion

While measures of inequality and concentration are often employed to assess heterogeneity, their underlying value functions may introduce features that are not well-suited for capturing heterogeneity as a distributional property. To address this limitation, I characterize a new family of measures—Descriptive Units of Heterogeneity (DUH)—that evaluate heterogeneity differently from the standard repurposed measures.

Building on the axiomatic foundations of classical measures such as the Gini coefficient, the Herfindahl–Hirschman Index, and Shannon's entropy, DUH are uniquely characterized by their sensitivity to two components: the size of the largest group and the evenness among minority groups. This structure departs from symmetric algebraic treatments of all group shares and allows for a more flexible and interpretable representation of heterogeneity. DUH support cardinal interpretation of changes, including proportional statements such as "system A is x times more heterogeneous than system B," through the principle of proportional transfers.

An essential part of this explication is the importance of researcher discretion in defining the group labels θ. While the overall model may appear elaborate, it provides a transparent and unambiguous framework for comparing heterogeneity across systems. As demonstrated in Section 5.2, DUH is not intended to replace existing measures but to complement them in contexts where sensitivity to minority group dynamics and zero-group non-invariance are analytically relevant. In settings where the focus is on dominant groups alone, such sensitivity may be undesirable. Nonetheless, DUH offers a tractable and axiomatic approach for researchers seeking to understand the evolution of heterogeneity in diverse empirical environments.

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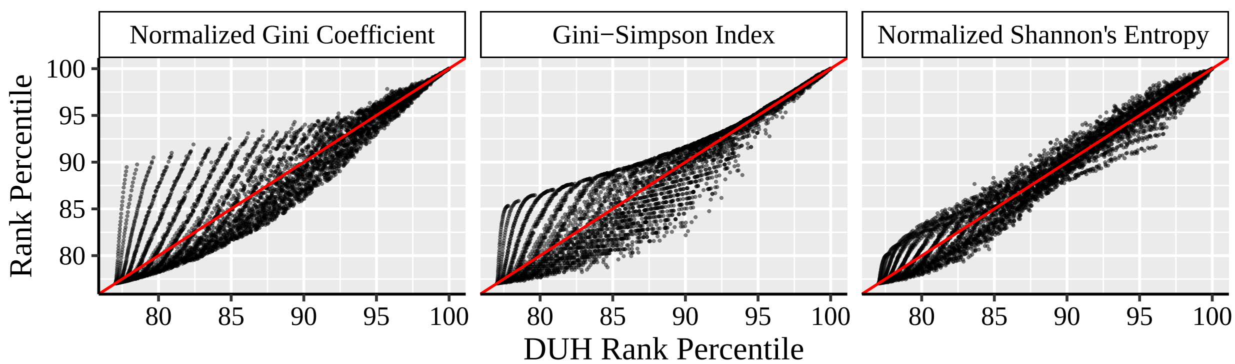
Appendices

Appendix A Additional Graphs and Tables

Table A1: Correlation Matrices with $|\theta| = 10$, $|\mathbb{P}| = 100$

	Correlation Matrix				Rank Correlation Matrix			
	DUH	Gini	GSI	SE	DUH	Gini	GSI	SE
DUH	1	0.910	0.760	0.888	1	0.933	0.940	0.979
Gini		1	0.559	0.688		1	0.992	0.957
GSI			1	0.947			1	0.964
SE				1				1

Figure B1: Rank Correlation for $|\theta| = 10$, $|\mathbb{P}| = 100$



Appendix B Proofs

Lemma 1: Any measure $\Phi(n_1, \ldots, n_G)$ of system $S = (n_1, \ldots, n_G)$ satisfies SYM if $(n_1, \ldots, n_G)$ is a vector ordered such that $n_1 \geq n_2 \geq \cdots \geq n_G$.

Proof of Lemma 1:

The proof is trivial, given that the groups are ordered by size and not the label of the groups. This is a convenient consequence of defining systems as mappings from the universe of groups to a vector of numbers.

Lemma 2: If an index $\Phi(\varphi, \psi)$ of heterogeneity satisfies SYM, INV, and PPT, it is monotonically decreasing in P_1, and therefore a positive monotonic transformation of $\frac{1}{P_1}$.

Proof of Lemma 2:

Take any 2 systems of G groups $S = (n_1, n_2, \ldots, n_G)$ and $S' = (n_1', n_2', \ldots, n_G')$ such that $\Phi(S) > \Phi(S')$ and that the $(n_2, \ldots, n_G) = \tilde{S} = \lambda \cdot \tilde{S}' = \lambda \cdot (n_2', \ldots, n_G')$, $\lambda \in \mathbb{R}_{++}$, then by scale invariance:

$$\Phi(n_1, n_2, \ldots, n_G) > \Phi(n_1', n_2', \ldots, n_G') = \Phi\left(n_1' \cdot \frac{n_S}{n_S'}, n_2' \cdot \frac{n_S}{n_S'}, \ldots, n_G' \frac{n_S}{n_S'}\right).$$

By the principle of proportional transfers, since $n_1 + n_2 + \cdots + n_G = n_1' \frac{n_S}{n_S'} + n_2' \frac{n_S}{n_S'} + \cdots + n_G' \frac{n_S}{n_S'}$,

$$\Phi(S) > \Psi(S') \iff n_1 < n_1' \cdot \frac{n_S}{n_S'} \iff P_1 < P_1'.$$

$\square$

Lemma 3: If an index $\Phi(\varphi, \psi)$ of heterogeneity satisfies SYM, INV, PPT, and IND, then $\Phi = \varphi \cdot \psi$.

Proof of Lemma 3:

Notice first that independence trivially implies that φ and ψ must be separable. Take any system S with G groups. By the principle of proportional transfers, it must be that $\forall \alpha \in \left[1, \frac{n_1 + \tilde{P}_2 \cdot n_1}{n - 2 + \tilde{P}_2 \cdot n_1}\right]$

$$\alpha \cdot \Phi(P_1, P_2, \ldots, P_G) = \Phi\left(P_1^\alpha, P_2 + \tilde{P}_2\left(P_1 - P_1^\alpha\right), \ldots, P_G + \tilde{P}_G\left(P_1 - P_1^\alpha\right)\right)$$

$$\iff \alpha \Phi(P_1, P_2, \ldots) = \Phi\left(P_1^\alpha, P_2', \ldots, P_G'\right) \iff \alpha = \frac{\varphi\left(P_1^\alpha\right)}{\varphi(P_1)}.$$

where $\exists \lambda \in \mathbb{R}_{++}\ s.t.\ \lambda P_g = P_g',\ \forall g \in \{2, \ldots, G\}$.

$\square$

Definition 10. *Take $p \in [1, \infty)$, $c_1 \in \mathbb{R}$, and $c_2 \in \mathbb{R}_{++}$. The function $\psi_p : \mathbb{R}_+^{G-1} \to \mathbb{R}$ defined as:*

$$\psi_p(S) = \psi\left(\tilde{P}_2, \ldots, \tilde{P}_G\right) = 1 - \left(\sum_{g=2}^{G} \left|\tilde{P}_g - \frac{1}{G-1}\right|^p\right)^{\frac{1}{p}},$$

is a measure of evenness in the distribution of minority groups.

Proposition 1. *Let $S^{\theta}_{\mathbb{P}}$ be an arbitrary system with more than three groups. Consider a measure of heterogeneity $\Phi_p\left(\hat{S}^{\theta}_{\mathbb{P}}\right) = \Phi\left(\phi\left(\hat{S}^{\theta}_{\mathbb{P}}\right), \psi_p\left(\hat{S}^{\theta}_{\mathbb{P}}\right)\right)$ that satisfies GSYM, INV, and IND. Holding $\hat{P}_1$ constant, if Φ is strictly increasing in ψ_p, then $\Phi\left(\phi\left(\hat{P}_1\right), \psi_p\right)$ satisfies PT if and only if $p > 1$.*[10]

Proof of Proposition 1:

Consider two ordered systems $S = (P_1, \dots, P_g, P_{g+1}, \dots, P_G)$ and $S' = (P_1, \dots, P_g - c, P_{g+1} + c, \dots, P_G)$ where $c < \frac{P_g - P_{g+1}}{2}$. Define $\tilde{c} = \frac{c}{P_2 + \cdots + P_G}$. I want to show that $\psi(S) < \psi(S')$ and $\Phi(S) < \Phi(S')$, thus satisfying *PT*.

Given S and S', we have

$$\psi_p(S) = 1 - \left(\left|\tilde{P}_2 - \frac{1}{G-1}\right|^p + \cdots + \left|\tilde{P}_g - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} - \frac{1}{G-1}\right|^p + \cdots + \left|\tilde{P}_G - \frac{1}{G-1}\right|^p\right)^{\frac{1}{p}}$$

$$\psi_p(S') = 1 - \left(\left|\tilde{P}_2 - \frac{1}{G-1}\right|^p + \cdots + \left|\tilde{P}_g - \tilde{c} - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} + \tilde{c} - \frac{1}{G-1}\right|^p + \cdots + \left|\tilde{P}_G - \frac{1}{G-1}\right|^p\right)^{\frac{1}{p}}.$$

Observe that

$$\begin{aligned}
&\psi_p(S) < \psi_p(S') \\
\iff &\left(\tilde{P}_g - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} - \frac{1}{G-1}\right|^p + C\Bigg)^{\frac{1}{p}} > \left(\tilde{P}_g - \tilde{c} - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} + \tilde{c} - \frac{1}{G-1}\right|^p + C\Bigg)^{\frac{1}{p}} \\
\iff &\left|\tilde{P}_g - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} - \frac{1}{G-1}\right|^p > \left|\tilde{P}_g - \tilde{c} - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} + \tilde{c} - \frac{1}{G-1}\right|^p.
\end{aligned}$$

Case 1: $\frac{1}{G-1} < \tilde{P}_{g+1} < \tilde{P}_g$, then

$$\begin{aligned}
&\left|\tilde{P}_g - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} - \frac{1}{G-1}\right|^p > \left|\tilde{P}_g - \tilde{c} - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} + \tilde{c} - \frac{1}{G-1}\right|^p \\
\iff &\left(\tilde{P}_g - \frac{1}{G-1}\right)^p + \left(\tilde{P}_{g+1} - \frac{1}{G-1}\right)^p > \left(\tilde{P}_g - \tilde{c} - \frac{1}{G-1}\right)^p + \left(\tilde{P}_{g+1} + \tilde{c} - \frac{1}{G-1}\right)^p \\
\iff &\frac{\left(\tilde{P}_g - \frac{1}{G-1}\right)^p + \left(\tilde{P}_{g+1} - \frac{1}{G-1}\right)^p}{2} > \frac{\left(\tilde{P}_g - \tilde{c} - \frac{1}{G-1}\right)^p + \left(\tilde{P}_{g+1} + \tilde{c} - \frac{1}{G-1}\right)^p}{2}. \\
\iff &p > 1 (\text{ making the function } x^p \text{ convex}).
\end{aligned}$$

Case 2: $\tilde{P}_{g+1} < \frac{1}{G-1} < \tilde{P}_g$, then

$$\begin{aligned}
&\left|\tilde{P}_g - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} - \frac{1}{G-1}\right|^p > \left|\tilde{P}_g - c - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} + c - \frac{1}{G-1}\right|^p \\
\iff &\left(\tilde{P}_g - \frac{1}{G-1}\right)^p + \left(\frac{1}{G-1} - \tilde{P}_{g+1}\right)^p > \left(\tilde{P}_g - c - \frac{1}{G-1}\right)^p + \left(\frac{1}{G-1} - \tilde{P}_{g+1} - c\right)^p \\
\iff &\underbrace{\left(\tilde{P}_g - \frac{1}{G-1}\right)^p - \left(\tilde{P}_g - c - \frac{1}{G-1}\right)^p}_{>0} + \underbrace{\left(\frac{1}{G-1} - \tilde{P}_{g+1}\right)^p - \left(\frac{1}{G-1} - \tilde{P}_{g+1} - c\right)^p}_{>0} > 0.
\end{aligned}$$

[10]This proposition suggests that we need to be careful regarding the functional form of $\phi\left(\hat{P}_1\right)$. To satisfy PT, it must be that when there is a transfer from the largest group to a minority group, the increase in $\phi\left(\hat{P}_1\right)$ must dominate the decrease in ψ_p in the case where evenness decreases. Also, if $G = 3$, this proposition holds with $p \geq 1$.

Case 3: $\tilde{P}_{g+1} < \tilde{P}_g < \frac{1}{G-1}$, then

$$
\begin{aligned}
&\left|\tilde{P}_g - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} - \frac{1}{G-1}\right|^p > \left|\tilde{P}_g - c - \frac{1}{G-1}\right|^p + \left|\tilde{P}_{g+1} + c - \frac{1}{G-1}\right|^p \\
\iff &\left(\frac{1}{G-1} - \tilde{P}_g\right)^p + \left(\frac{1}{G-1} - \tilde{P}_{g+1}\right)^p > \left(\frac{1}{G-1} - \tilde{P}_g + c\right)^p + \left(\frac{1}{G-1} - \tilde{P}_{g+1} - c\right)^p \\
\iff &\frac{\left(\frac{1}{G-1} - \tilde{P}_g\right)^p + \left(\frac{1}{G-1} - \tilde{P}_{g+1}\right)^p}{2} > \frac{\left(\frac{1}{G-1} - \tilde{P}_g + c\right)^p + \left(\frac{1}{G-1} - \tilde{P}_{g+1} - c\right)^p}{2}. \\
\iff &p > 1 (\text{ making the function } x^p \text{ convex}).
\end{aligned}
$$

Last, if $G = 3$, then case 2 is the only case, meaning that the RHS of the statement can be expanded to $p \geq 1$. □

Proposition 2. *If an index $\Phi(\varphi, \psi)$ of heterogeneity satisfies INV, IND, and PPT, then it must be $\Phi = \varphi(P_1) \cdot \psi(\tilde{P}_2, \ldots, \tilde{P}_G)$ where $\varphi(P_1) = -c \cdot log_q(P_1)$, $c \in \mathbb{R}_{++}$.*

Proof of Proposition 2:

From the previous two lemmas, we know that $\varphi(P_1)$ must be a positive monotonic transformation of $\frac{1}{P_1}$ and that for α such that $P_1^\alpha > P_2 + \tilde{P}_2(P_1 - P_1^\alpha)$, we must have

$$\alpha = \frac{\varphi(P_1^\alpha)}{\varphi(P_1)}.$$

Notice that the only positive monotonic transformation that would satisfy this is $log_q\left(\frac{1}{P_1}\right)$, up to a positive scalar multiplication. Further, notice that any $log_q\left(\frac{1}{P_1}\right)$ can be rewritten as $\frac{ln\left(\frac{1}{P_1}\right)}{ln(q)}$, so it is equivalent to write $c \cdot ln\left(\frac{1}{P_1}\right)$. As such, $\varphi(P_1) = c \cdot ln\left(\frac{1}{P_1}\right)$, $c \in \mathbb{R}_{++}$ is the unique function, up to positive scalar multiplication, of majority proportions that can lead to $\Phi(\varphi, \psi)$ satisfying independence, scale invariance, and the principle of proportional transfers. □

Theorem: The descriptive units of heterogeneity, Φ defined as

$$
\begin{aligned}
\Phi_p(n_1, \ldots, n_G) &= -\frac{ln\,(P_1)}{ln(G)}\left[1 - \left(\sum_{g=2}^{G}\left(\tilde{P}_g - \frac{1}{G-1}\right)^p\right)^{\frac{1}{p}}\right] \\
&= \frac{ln\,(P_1)}{ln(G)}\left[\left(\sum_{g=2}^{G}\left(\tilde{P}_g - \frac{1}{G-1}\right)^p\right)^{\frac{1}{p}} - 1\right], \\
\text{where } P_1 &= \frac{n_1}{n_1 + \cdots + n_G},\ \tilde{P}_g = \frac{n_g}{n_2 + \cdots + n_g},\ p \in (1, \infty).,
\end{aligned}
$$

constitute a uniquely determined family of measures—up to positive scalar multiples—that incorporate evenness through the function ψ_p, and satisfy the axioms of group symmetry, scale invariance, independence, the principle of transfers, the principle of proportional transfers, and contractibility.

Proof of Theorem 1:

Propositions 1 and 2 combined implies that DUH satisfy SYM, INV, PPT, and IND, but not necessarily PT. I have only shown that DUH satisfy PT when either φ or ψ_p is held constant, which means that I need to be careful with the functional form of $\varphi(P_1)$. To satisfy *PT* overall, it must be that when there is a transfer from the majority group to the minority group, the increase in Φ through $\varphi(P_1)$ dominates the decrease in Φ through ψ_p in the case in which evenness decreases as a result of the transfer.

To show that Φ satisfies PT, we only need to look at the extreme case in which P_1 is close to 1 and $\psi_p = 1$. In this case, a simple transfer from n_1 to n_2 will decrease ψ_p the most. For simplicity, we will consider the case in which $p = 2$, so that Φ is simply

$$\Phi(n_1, \ldots, n_G) = \frac{ln\,(P_1)}{ln(G)} \left[\sqrt{\sum_{g=2}^{G} \left(\tilde{P}_g - \frac{1}{G-1} \right)^2} - 1 \right].$$

Denote $n_2 + \cdots + n_G$ as $\tilde{n}_S$, a transfer of x from n_1 to n_2 when $\psi = 1$ can be written as

$$\Phi_2 = ln \left(\frac{n_1 - x}{n_S} \right) \left[\sqrt{ \left(\frac{\frac{\tilde{n}_S}{G-1} + x}{\tilde{n}_S + x} \right) + (G-2) \left(\frac{\frac{\tilde{n}_S}{G-1}}{\tilde{n}_S + x} - \frac{1}{G-1} \right)^2 } - 1 \right].$$

Taking the derivative of this expression with respect to x, we have $\forall x \in \left[0, \frac{(G-1)n_1 - \tilde{n}_S}{G}\right]$:

$$\frac{d}{dx}\Phi_2(n_1 - x, n_2 + x, n_3, \ldots, n_G) = \frac{\sqrt{\frac{G-2}{G-1}} \left[\tilde{n}_S (x - n_1) ln \left(\frac{n_1 - x}{n_S} \right) + x(b + x) \right]}{(x - n_1)(x + \tilde{n}_S)^2} + \frac{1}{n_1 - x} > 0.$$

As such, the Family DUH satisfies PT, in addition to GSYM, INV, IND, and PPT. To see that the family DUH satisfies CON, see that P_1 and all the $\tilde{P}_g$'s are invariant to the addition of zero-groups, so I only need to check that ψ_p decreases when G increases by 1. In other words, I only need to show that $\left(\sum_{g=2}^{G} \left| \tilde{P}_g - \frac{1}{G-1} \right|^p \right)^{\frac{1}{p}}$ increases in G. First, observe the following inequality:

$$\begin{aligned}
\sum_{g=2}^{G} \left| \tilde{P}_g - \frac{1}{G-1} \right|^p &= \sum_{g=2}^{G} \left| \tilde{P}_g - \frac{1}{G} + \frac{1}{G} - \frac{1}{G-1} \right|^p \\
&\leq \sum_{g=2}^{G} \left(\left| \tilde{P}_g - \frac{1}{G} \right|^p + \left| \frac{-1}{G(G-1)} \right|^p \right) = \sum_{g=2}^{G} \left| \tilde{P}_g - \frac{1}{G} \right|^p + \frac{G-1}{G^p (G-1)^p} \\
&< \sum_{g=2}^{G} \left| \tilde{P}_g - \frac{1}{G} \right|^p + \frac{1}{G^p} = \sum_{g=2}^{G} \left| \tilde{P}_g - \frac{1}{G} \right|^p + \left| 0 - \frac{1}{G} \right|^p = \sum_{g=2}^{G+1} \left| \tilde{P}_g - \frac{1}{G} \right|^p
\end{aligned}$$

Taking the p^{th} root on both side, we have

$$\left(\sum_{g=2}^{G}\left|\tilde{P}_g - \frac{1}{G-1}\right|^p\right)^{\frac{1}{p}} < \left(\sum_{g=2}^{G+1}\left|\tilde{P}_g - \frac{1}{G}\right|^p\right)^{\frac{1}{p}}. \quad \square$$

To see that the family DUH satisfies UNI, verify that $\Phi(1,0,\ldots,0) = 0$ and $\Phi\left(\frac{1}{G},\ldots,\frac{1}{G}\right) = 1$, for all $G \in \mathbb{N} \setminus \{1,2\}$. $\square$

Appendix C Characterization Axioms of HHI and SE

Axiom C.1 (Expandability (EXP))**.** *Take an arbitrary system $S_{\mathbb{P}}^{\theta} = (n_1, \ldots, n_G)$ of G groups.Φ satisfies expandability if*

$$\Phi(n_1, \ldots, n_G) = \Phi(n_1, \ldots, n_G, 0).$$

It should be clear that neither EXP nor CON attempts to pin down the functional form of a meaure. Rather, these two opposing axioms serve as the divide between a unit for concentration and a unit for heterogeneity.

As discussed extensively by Atkinson (1970), the fact that PDT only induces partial ordering implies that specific functional forms can always be chosen to induce different total orders when neither system's distribution second-order stochastically dominates the other.[11] The functional forms of both HHI and SE were able to be uniquely characterized because of this feature.[12] HHI uses EXP and the replication principle (REP), and SE uses EXP and Shannon's additivity (SADD).

> **[REP] Replication Principle.** $\Phi(n_1, \ldots, n_G)$ satisfies the replication principle for concentration if replicating a system k times divides the system concentration by k.

Axiom C.2 (Replication Principle (REP))**.** *Take an arbitrary system $S_{\mathbb{P}}^{\theta}$ of G groups. Φ satisfies the replication principle for concentration if replicating a system k times divides the system concentration by k.*

For example, take $k \in \mathbb{N}$,

$$\frac{1}{k}\Phi(n_1, \ldots, n_G) = \Phi\left(\underbrace{\frac{n_1}{k}, \frac{n_1}{k}, \ldots, \frac{n_1}{k}}_{\text{Sum to } n_1}, \frac{n_2}{k}, \frac{n_2}{k}, \ldots, \underbrace{\frac{n_G}{k}, \ldots, \frac{n_G}{k}}_{\text{Sum to } n_G}\right).$$

REP pins down the cardinal meaning of the unit by linking the multiplication of the unit to how many times a system is divided/replicated into a system with more groups. Chakravarty and Eichhorn (1991) show that if a concentration unit C can be represented as a self-weighted quasilinear mean, then C is the Hannah-Kay index of concentration if and only if C satisfies the fundamental axioms and REP.[13] Since HHI is $H_{\alpha=2}^{n}(S)$, it is the unique self-weighted quasilinear concentration unit that satisfies the fundamental axioms, EXP, and REP. Notice that REP applies only when *every group* is broken into multiple groups. To prescribe how the index behaves when only one group is broken up, SE forgoes REP and restricts the measure based on Shannon's additivity instead.

[11] Newbery (1970) shows that no additive functional forms can be chosen to induce the same order as the Gini coefficient. This impossibility theorem stems from and reaffirms the point made by Atkinson (1970).

[12] Similarly, I use this feature to uniquely characterize the descriptive units of heterogeneity.

[13] A relative concentration index $C : D \to \mathbb{R}$ is called a self-weighted quasilinear mean if for all $n \in \mathbb{N}$, $x \in D^n$, $C^n(x)$ is of the form

$$C^n(x) = \phi^{-1}\left[\sum_{g=1}^{G} P_g \phi\left(P_g\right)\right],$$

where $\phi : (0, 1] \to \mathbb{R}$ is strictly monotonic.

Axiom C.3 (Shannon's additivity (SADD))**.** *Define* $n_{gj} \geq 0$ *such that* $n_g = \sum_{j=1}^{m_g} n_{gj}$*,* $\forall g \in \{1, \ldots, G\}$ *and* $\forall j \in \{1, \ldots, m_g\}$*. The measure* Φ *satisfies Shannon's additivity if*

$$\Phi(n_{11}, \ldots, n_{Gm_G}) = \Phi(n_1, \ldots, n_G) + \sum_{g=1}^{G} \frac{n_g}{n_S} \cdot \Phi\left(\frac{n_{g1}}{n_g}, \ldots, \frac{n_{gm_g}}{n_g}\right).$$

By setting $m_g = 1$, $\forall g \in \{1, \ldots, G-1\}$ and $n_{G'} = n_G + n_{G+1}$), SADD implies htat

$$\Phi(n_1, \ldots, n_G, n_{G+1}) = \Phi(n_1, \ldots, n_{G'}) + \frac{n_{G'}}{n_1 + \ldots + n_{G-1} + n_{G'}} \cdot \Phi\left(\frac{n_G}{n_{G'}}, \frac{n_{G+1}}{n_{G'}}\right).$$

SADD pins down how the decomposition of group(s) in a system should influence the unit.[14]

For detailed proofs of the unique characterization of SE and explanations of SADD, readers should refer to Suyari (2004) and Chakrabarti, Chakrabarty et al. (2005).

[14]An example of decomposing a group is to split the sales of Macs into Mac desktops and Mac laptops, for the purpose of measuring the heterogeneity of Apple's revenue streams.