Michigan State EC812B Organized Lecture Notes

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If I happen to be your TA for this course, please make sure to use this document at your own risk. Mistakes on your exams caused by contents here will still count against you.[1]*. It looks nicer if I have a full paragraph, so here is lorem ipsum.*

Contents

[1] That being said, please let me know if you come across any mistakes/typos in here so I can update it. Last updated: April 29, 2025.

What is game theory?

In 812A, we studied how agents behave in competitive markets using the consumer demand framework. For the most part, we have studied agents who are similar, if not identical, and our investigations have been mostly at the infinitely many consumers/firms setting. But what if, surprisingly, those assumptions that we have be operating under are unrealistic? What is markets are not actually perfectly competitive?

This is where game theory comes in. Through the study of games, we can try to understand what would happen under different market environments.

Now, recall that the assumption of a perfectly competitive market generally involves 3 components:

1. Negligibly Small Externalities in Actions
2. No Agent has Dictatorial Market Power (i.e., all agents are price-taker)
3. All Agents Have Perfect Information

When we relaxes either the first or the second component, there are still some "tricks" we could use to study the market in the general equilibrium framework[2]. So for the most part, we will focus on how studying how *information asymmetry* in a market distorts our GE results. Later on, with what we will have learned, we can take a peek into the world of **Mechanism Design**, which involves designing games/interactions for agents in order to improve on said inefficiency[3]. The new frontier (circa 2023) of mechanism design is **Information Design**, which sadly, will not be covered in this class.

Let's Begin!

1 Introduction to (Non-Cooperative) Game Theory

In Game Theory, we want to study how agents behave in the absence of binding contracts, since the existence of a binding contract in one market would directly tell you what is supposed to happen in that market[4].

[2]Such as Coase Theorem for internalizing externalities and the studies of Monopolistic/Oligopolistic markets.

[3]Essentially, Game Theory and Mechanism Design are two sides of the same coin. The studying of one leads to the other.

[4]As an aside, cooperative Game Theory is a real thing and is mostly about what happens when the contracts are broken. We will not cover that in this course

What is a game? A game is a formal representation of *interaction(s)* between multiple *rational* agents in a setting of *strategic interdependence*. i.e., what would agent A do when agent B does something. Economists use games as microcosms of market environments with infinitely many agents in order to study the effect of information asymmetry.

In general, a game has 3.5 elements in its environment:

1. Players/Agents
2. Rules/Actions: This is something that was absent/unspecified in the structure of general equilibrium framework. We can use this to study how agents *dynamically adjust*. In GE, we only see the end results, but having rules can better help us understand how we got there.
3. (a) Outcomes: Is it a repeated game? If it an infinite game? Does player 1 act after player 2 has acted according to player 1's last move?
   (b) Payoffs: Utility representation of outcomes. Players use the payoffs to rank their preferences over actions.

Example: Standard Matching Pennies Game

Consider the game where 2 players **simultaneously** choose to either place a coin with *Heads* or *Tails* up. Player 1 wins if the sides do *not* match, and player 2 wins otherwise. Using the above framework, we know the game can be represented by

1. Players: $I = \{P_1, P_2\}$
2. Actions: $A_1 = \{H_1, T_1\}$, $A_2 = \{H_2, T_2\}$
3. Outcomes: $Z = \{(H_1, H_2), (T_1, T_2), (H_1, T_2), (T_1, H_2)\}$
4. Payoffs: $U = \{u_1(H_1, H_2) = u_1(T_1, T_2) = -1,\ u_1(H_1, T_2) = u_1(T_1, H_2) = 1,\ u_2 = -u_1\}$

Notice that in every single outcome, $u_1 + u_2 = 0$. This is the (in)famous zero-sum game. Additionally, if one think about this from a GE standpoint, it means that every outcome in this game is efficient.

Example: Meeting in New York (Focal Point)

Consider the game where Thomas and Schelling need to figure out where to meet up in NYC. They had mentioned that they would meet at either the Grand Central Station or the Empire State Building at a given time, but they have since lost touch and have no way of contacting the other person. Now it's almost time for them to meet, and they each get a payoff of 1 if they successfully meet up, and 0 otherwise. Alternative to the last example, we can represent this game with a table of payoffs:

Thomas \ Schelling	GCS	ESB
GCS	$1, 1$	$0, 0$
ESB	$0, 0$	$1, 1$

In this setting, we call Thomas the **Row Player** of this game, and Schelling the **Column Player**. Additionally, notice that this is no longer a zero-sum game, and only the outcomes (GCS,GCS) and (ESB,ESB) are efficient outcomes.

Hopefully, at this point, you have realized that you can represent every game in different forms. In practice, certain games are more easily represented by one form over the other, and as you learn more and more problems, you will likely develop your own preference. To get into this, know that the representation in the first example (standard matching pennies) uses an **Extensive Form Representation** and the second example (Meeting in New York) uses the **Norm Form Representation**.

1.1 Extensive Form Representation (Game Tree)

Extensive forms can be presented in 2 ways: mathematical form (the standard matching pennies example) and game tree (see figure 1). In general, extensive forms represents **sequential games** better, as drawing a tree can implicitly represent the *order* actions happen.

Consider a **sequential version** of the standard matching pennies game with which we are now all too familiar. Player 1 moves first, player 2 observes, then player 2 moves, and the rest of the rules are the same. We can represent this game in Figure 1.1.1 and the sequential aspect of this game will be directly embedded in the tree.

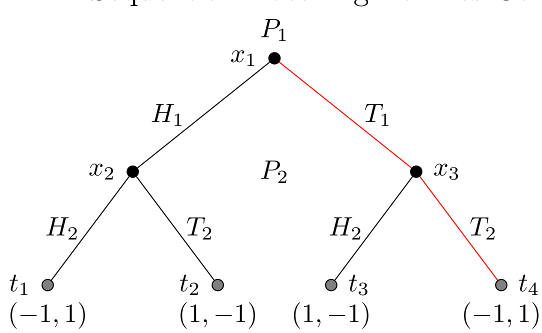
There are a couple things you need to know about the game tree:

- Each game tree starts at one initial node (also called the root)
- Each choice is represented by a branch
- There is always an *unique path* from the initial node to any other node
- The nodes at the end are called terminal nodes
- All nodes that are not terminal nodes are called decision nodes

In Figure 1.1.1,

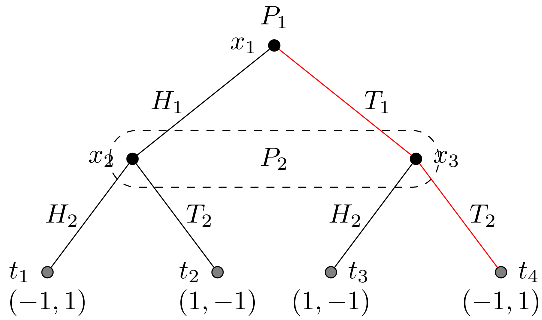
- The node accompanied by P_1 is the *initial node*
- The lines with H_1, T_1, H_2, T_2 are *branches*
- The red line from the initial node to the terminal node is an *unique path*
- The black nodes that capture P_1 and P_2 are *decision nodes*
- The bottom 4 gray nodes are *terminal nodes*.

Figure 1.1.1: Sequential Matching Pennies Game Tree



Now consider a modification of this game where P_2 does not observe what P_1 plays. We can represent such game in a similar tree. In Figure 1.1.2, we have a game tree that looks almost exactly the same as Figure 1.1.1, except we need some way to represent the fact that P_2 has the same information (not knowing whether P_1 played H_1 or T_1). We "group" the two P_2 nodes with a dotted eclipse to represent that these two nodes have the exact same information.

Figure 1.1.2: Sequential Matching Pennies Game Tree



If we really think about it, what is the difference between a simultaneous matching game and a sequential game with no observation? Nothing! Since Player 2's does not have any information on what Player 1's move is, these two games are identical. This thought experiment tells us that *The timing of the game (static v. dynamic) does not affect the structure of the games; rather the information sets each player posses at the time of action does.*
So our logical next step is to study games of **Perfect** vs. **Imperfect Information**.

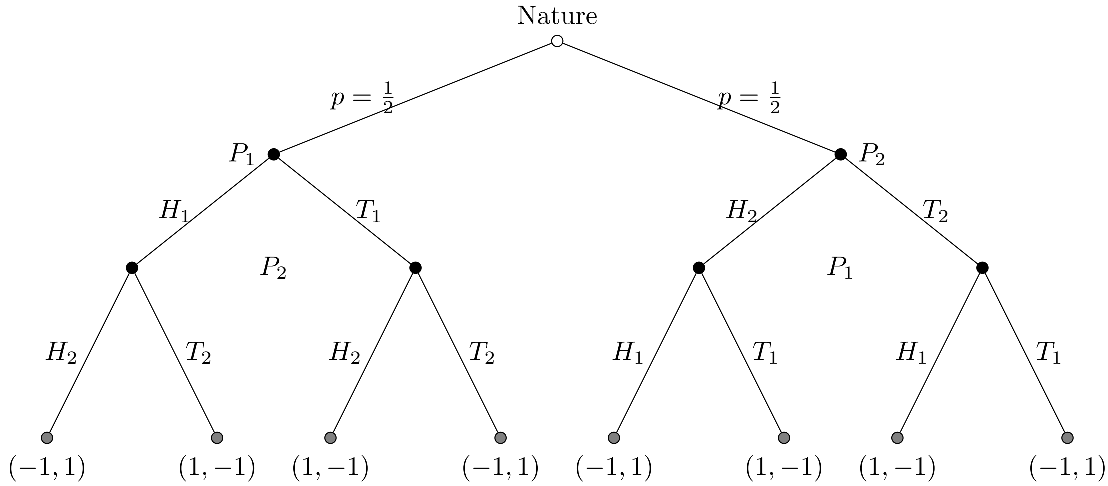
- Perfect Information: All players know all decisions of all other players who moved earlier in the game. (e.g., tic-tac-toe)
- Imperfect Information: Otherwise (e.g., simultaneous matching pennies game)

Since there is alwasy a unique path from the initial node to a decision node, knowing the node a player is at tells us the past history of the game, i.e., information a player has at that point. More importantly,

- A decision node captures the history of plays by a player and their opponent
- If an information set includes multiple nodes, the game is of *imperfect information*

Consider a sequential matching pennies game where the order the players move is randomized. When a move has uncertainty, we represent it with a hollowed node and the player is denoted **Nature**. In practice, when a node is mapped to a decision with uncertainty, we represent it with a hollowed node.

Figure 1.1.3: Sequential Matching Pennies Game Tree with Random Order

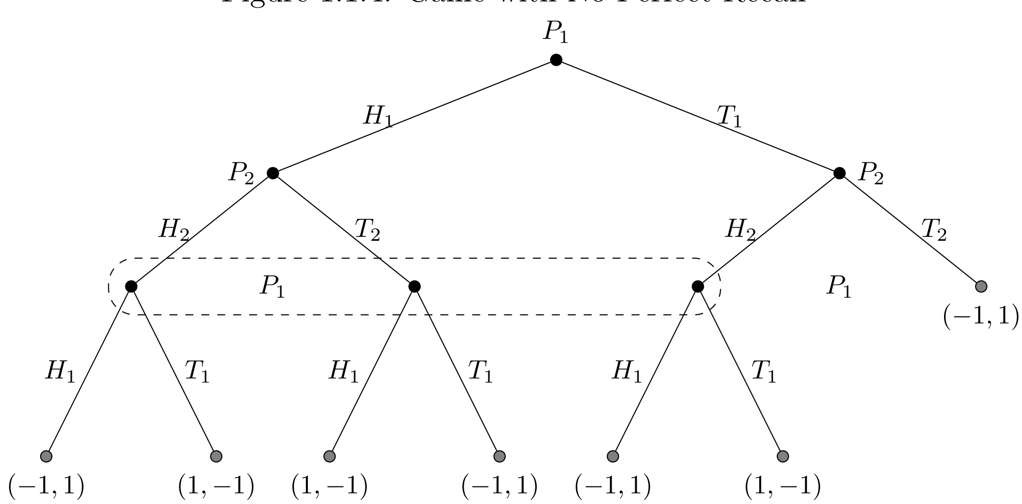


1.1.1 Assumption of Perfect Recall

The assumption of perfect recall states that *every player can memorize all of their own previous actions along with the actions of other players that they have observed*. For example, Figure 1.1.4 represents a game where player 1 does not have perfect recall.

Consider a sequential matching pennies game where if 2 tails are played, then player 1 gets to change what she plays. However, player 1 would only know whether she gets to change their choice, but not what the other player played. This game can be represented by the following tree:

Figure 1.1.4: Game with No Perfect Recall



In this game, player 1 conveniently "forgets" what she played at the beginning. Even though we, as the outsider, know that if player 1 gets to choose again, she should change her choice to heads because if they got to choose again it must have been that she chose tails first and player 2 matched it. However, since player 1 does not have perfect recall, they do not have our insight.

1.1.2 Assumption of Common Knowledge

At a glance, this may sound like a silly assumption. However, this assumption is actually quite critical to our study of games. The assumption of common knowledge is about assuming what other players know, and it can have infinitely many layers. The assumption goes:

Level 0:	All players know the structure of the game
Level 1:	All players know that all other players know the structure of the game
Level 2:	All players know that all other players know that all other players know the structure of the game
	⋮
Level n:	All players know (that all other players know)^n the structure of the game

1.2 Extensive Form Representation (Mathematical)

A game in extensive form is mathematically specified by:

$\Gamma_\varepsilon = \{\mathcal{I}, \mathcal{A}, \mathcal{X}, p(\cdot), c(\cdot), \mathcal{H}, H(\cdot), \iota(\cdot), \rho(\cdot), u_i\}$

$\mathcal{I}$	The set of players $\{1, \ldots, i, \ldots, N\}$
$\mathcal{A}$	The set of actions of all players
$\mathcal{X}$	The set of nodes
T	Terminal nodes $T = \{x \in \mathcal{X} \mid s(x) = \varnothing\}$
$\mathcal{X} \setminus T$	Decision node. All nodes that are not terminal nodes
$p : \mathcal{X} \to \mathcal{X} \cup \varnothing$	The predecessor function that specifies the immediate predecessor of each node[5]
$s(x) = p^{-1}(x)$	are the immediate successor nodes of x
$c_i : \mathcal{X} \setminus T \rightrightarrows \mathcal{A}$	Choice mappings that specify the actions available to player i at their decision node $x \in \mathcal{X} \setminus T$
$\mathcal{H}$	A collection of information sets
$H : \mathcal{X} \setminus T \to \mathcal{H}$	Specifies an information set to each decision node[6]
$\iota : \mathcal{H} \to \mathcal{I} \cup \{0\}$	Assigns each information set in $\mathcal{H}$ to the player who moves at the decision node $H^{-1}(H)$

$\rho : \mathcal{H}_0 \times \mathcal{A} \to [0,1]$	The probability assignment to actions at information sets where nature moves. $\forall H \in \mathcal{H}_0,\ \rho(H,a) = 0$ if $a \notin C(H),\ \sum_{a \in c(H)} \rho(H,a) = 1$
$U : T^{\lvert\mathcal{I}\rvert} \to \mathbb{R}$	$U = \{u_1, \ldots, u_N\}$ which is the collection of functions assigning utilities to outcomes at terminal nodes.

For example, the standard matching pennies game in Figure 1.1.2 can be specified in the mathematical extensive form as

$\Gamma_\varepsilon = \{\mathcal{I}, \mathcal{A}, \mathcal{X}, p(\cdot), c(\cdot), \mathcal{H}, H(\cdot), \iota(\cdot), \rho(\cdot), u_i\}$

$\mathcal{I}$	$\{P_1, P_2\}$
$\mathcal{A}$	$\{H_1, T_1, H_2, T_2\}$
$\mathcal{X}$	$\{x_1, x_2, x_3, t_1, t_2, t_3, t_4\}$
T	$T = \{t_1, t_2, t_3, t_4\}$
$\mathcal{X} \setminus T$	$\mathcal{X} \setminus T = \{x_1, x_2, x_3\}$
$p : \mathcal{X} \to \mathcal{X} \cup \varnothing$	$p(t_1) = p(t_2) = x_2,\ p(t_3) = p(t_4) = x_3,\ p(x_2) = p(x_3) = x_1,\ p(x_1) = \varnothing$
$s(x) = p^{-1}(x)$	$s(x_1) = \{x_2, x_3\},\ s(x_2) = \{t_1, t_2\}, s(x_3) = \{t_3, t_4\}$
$c_i : \mathcal{X} \setminus T \rightrightarrows \mathcal{A}$	$c(x_1) = \{H_1, T_1\},\ c(x_2) = c(x_3) = \{H_2, T_2\}$
$\mathcal{H}$	$\mathcal{H} = \{I_1 = \{x_1\},\ I_2 = \{x_2, x_3\}\}$
$H : \mathcal{X} \setminus T \to \mathcal{H}$	$H(x_1) = I_1,\ H(x_2) = H(x_3) = I_2$
ι	$\iota(I_1) = P_1,\ \iota(I_2) = P_2$
$\rho : \mathcal{H}_0 \times \mathcal{A} \to [0,1]$	Not applicable because there is no nature
$U : T^{\lvert\mathcal{I}\rvert} \to \mathbb{R}$	$u_1(t_1) = u_1(t_4) = -1,\ u_1(t_2) = u_1(t_3) = 1,\ u_2 = -u_1$

1.2.1 Information in a Game

Definition (Information Set): A collection of nodes that a player cannot distinguish between. At every node in the same information set, the player must have the same set of possible action.

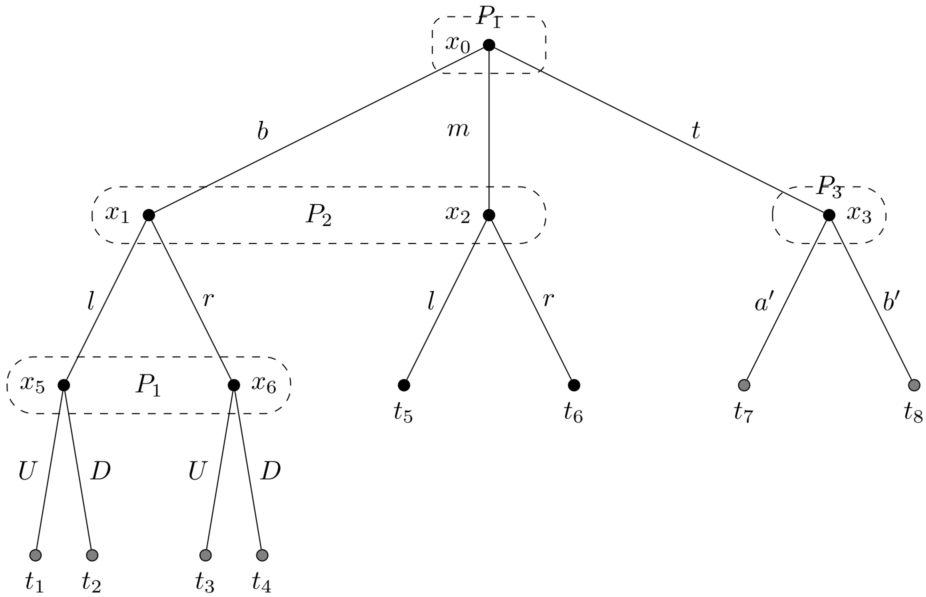
Formally, $\forall H \in \mathcal{H}$, let $c(H) \subset \mathcal{A}$ be the set of actions available to the player $\iota(H)$, then $\forall x, x' \in H$ such that $H(x) = H(x')$, it must also be that $c(\{x\}) = c(\{x'\})$.

[5] $p(x_0) = \varnothing$ where x_0 is the initial node. $p(x)$ is non-empty for all $x \in \mathcal{X} \setminus \{x_0\}$.

[6] Note that information sets are **partitions** of decision nodes since each node can only be in one information set.

Definition (Perfect Information): A game is of **perfect information** if all information sets contain exactly one node ($\forall H \in \mathcal{H}$, $|H| = 1$). Otherwise, it is a game of **imperfect information**.

Figure 1.2.1: Game of Imperfect Information



In this example, P_3 has perfect information but P_1 and P_2 do not. Overall, since there are 2 information sets with 2 nodes, this is a game of imperfect information.

1.3 Normal Form Representation

A normal form representation of a game is $\Gamma_N = (\mathcal{I}, (S)_{i\in\mathcal{I}}, (U)_{i\in\mathcal{I}})$ where $\mathcal{I}$ is the set of players, $(S)_{i\in\mathcal{I}}$ is a sequence of sets of *strategies*, and U is a sequence of *payoffs* $u_i = (s_1, \ldots, s_I)$.
Just as extensive form representation is good for capturing the order of actions in the game tree, normal form representation is good for simultaneous/static games.

1.3.1 Strategies

Definition (Strategy): A strategy is a complete contingent plan of a player that specifies the action of the player at every possible move. Formally, denote

$\mathcal{H}_i$	The collection of information sett of player i
$\mathcal{A}$	The set of all possible actions of all players
$c(H) \subset \mathcal{A}$	The set of actions available at information set $H \in \mathcal{H}_i$

A **strategy** is a function $s_i : \mathcal{H}_i \to \mathcal{A}$ such that $\forall H \in \mathcal{H}_i,\ s_i(H) \in c(H)$

Example:

Example 1 Meeting at NYC
Both players have 2 strategies: $\{GCS, ESB\}$

Example 2 Sequential Matching Pennies
$S_1 = \{(H_1), (T_1)\}$
$S_2 = \{(H_H, H_T), (H_H, T_T), (T_H, H_T), (T_H, T_T)\}$

It is important to know that a strategy specifies actions for a player *even at information sets that are not reached during the game*[7]. This means that a strategy includes actions that are made irrelevant by the player's own plan. In Figure 1.2.1, player 1 has $3 \times 2 = 6$ possible strategies: $S_1 = \{(b, U), (b, D), (m, U), (m, D), (t, U), (t, D)\}$).

A combination of all players' strategy **generates/induces** an outcome of the game. Once we specify the strategies and outcomes of the game, we have the so called **normal/strategic** form representation of the game.

Remark: An interpretation of strategy is *A set of complete instructions that can be perfectly carried out by a representative.* Another interpretation is that strategies can be thought of as *carrying out an action with probability 1*, but it does not mean that actions with probability 0 will not happen.

[7]Recall that strategies are mappings from the information sets to actions

Example: Normal Form Representation

Example 1: Sequential Matching Pennies

$\Gamma_N = (\mathcal{I}, (S_1, S_2), (u_1, u_2))$

$\mathcal{I} = \{1, 2\}$, $S_1 = \{H_1, T_1\}$, $S_2 = \{(H_H, H_T), (H_H, T_T), (T_H, H_T), (T_H, T_T)\}$

P_1 \ P_2	(H_H, H_T)	(H_H, T_T)	(T_H, H_T)	(T_H, T_T)
H_1	$-1, 1$	$-1, 1$	$1, -1$	$1, -1$
T_1	$1, -1$	$-1, 1$	$-1, 1$	$1, -1$

Example 2: Simultaneous Matching Pennies

$\Gamma_N = (\mathcal{I}, (S_1, S_2), (u_1, u_2))$

$\mathcal{I} = \{1, 2\}$, $S_1 = \{H_1, T_1\}$, $S_2 = \{H_2, T_2\}$

P_1 \ P_2	H_2	T_2
H_1	$-1, 1$	$1, -1$
T_1	$1, -1$	$-1, 1$

1.3.2 Mixed Strategies

Since a strategy can be interpreted as carrying out an action with probability 1, we can also think about scenarios where a player may want to carry out a set of actions with the total probability summed up to 1. We call these **Mixed Strategies**, as compared to the **Pure Strategies** where 1 action is carried out with probability 1. If a mixed strategy assigns non-zero probability to every single pure strategy, then it is called a **totally/strictly mixed strategy**.

Mathematically, a mixed strategy is a probability measure $\sigma_i : S_i \to [0, 1]$ that assigns probability over each terminal node and $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$. We denote *mixed strategies* in a normal form game as

$$\Gamma_N^{\Delta} = \{\mathcal{I}, (\Delta(S_i))_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}}\}$$

where the utility evaluation assumes the *Expected Utility Theory* such that

$$u_i(\sigma_1, \sigma_2, \ldots, \sigma_I) = \sum_{s\in S} [\sigma_1(s_1)\cdot\sigma_2(s_2)\cdots\sigma_I(s_I)]\cdot u_i(s_1,\ldots,s_I)$$

1.3.3 Behavior Strategies

Given an extensive form game, a behavior strategy for player i specifies a probability measure λ_H over possible actions at each information set $H \in \mathcal{H}_i$.

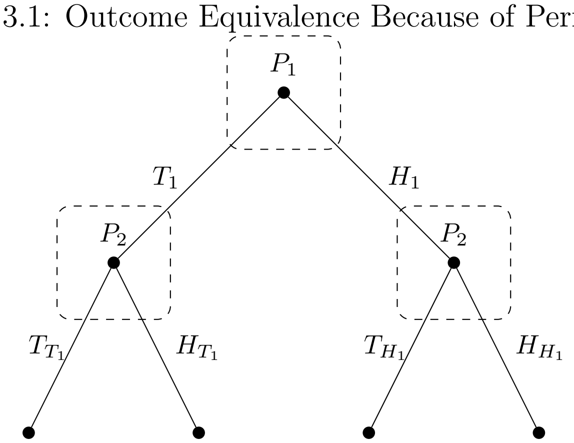
Definition (Outcome Equivalence:) Two strategies are said to be **outcome equivalent** if each outcome/terminal node has the same probability of being reached under the 2 strategies.

Proposition: If a game satisfies perfect recall, then all randomization of behavior strategies and randomization of pure strategies (i.e., mixed strategies) are outcome equivalent. (i.e., for any mixed strategy, there exists a behavior strategy that generates equivalent outcomes.)

Example: Example of Outcome Equivalence

Consider the following game represented by the game tree. The behavior strategy:
$B_2 = \{(\frac{1}{4}H_H + \frac{3}{4}T_H)(\frac{2}{3}H_T + \frac{1}{3}T_T)\}$.
This strategy has the outcome equivalent mixed strategy:
$M_2 = \{\frac{1}{4}H_H H_T + \frac{1}{4}\frac{1}{3}H_H T_T + \frac{3}{4}\frac{2}{3}T_H H_T + \frac{3}{4}\frac{1}{3}T_H T_T\}$

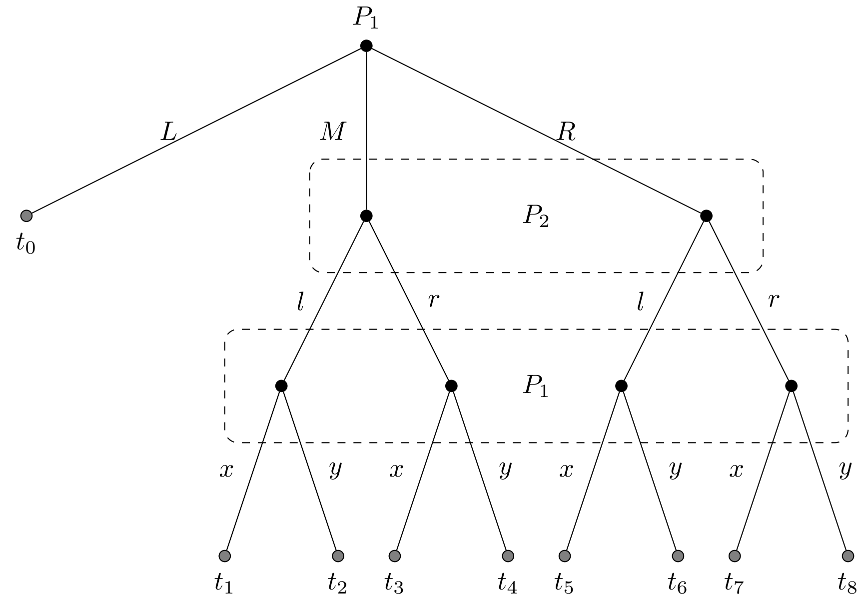
Figure 1.3.1: Outcome Equivalence Because of Perfect Recall



Example: Example of NO Outcome Equivalence

Consider the following game represented by the game tree and the mixed strategy: $M_1 = \{\frac{1}{2}Mx + \frac{1}{2}Ry\}$. Notice that there is no outcome equivalent behavior strategy, since $\sigma(Mx) = \frac{1}{2}$ means $\sigma(My) = 0$. So at the second information set, any behavior strategy must have $\lambda(y) = 0$. But $\sigma(Ry) = \frac{1}{2}$, meaning that at that same information set, any behavior strategy must have $\lambda(y) > 0$, which is where we reach a contradiction.

Figure 1.3.2: No Outcome Equivalence Because of Imperfect Recall



2 Solution Concepts

2.1 Dominance-Based Solution Concepts

Our central question in game theory is to *predict outcomes of a game played by rational players who are fully knowledgeable about the structure of the game.*

Take the simple case of a simultaneous-move game[8]. Naturally, there may be some strategies that are better than the others and should always be played. The ways of defining these strategies can be generally thought of as one of the following:

(i) Dominant/Dominated Strategies

(ii) Rationalizable Strategy Nash Equilibrium

(iii) Mixed Strategy Nash Equilibrium

(iv) Trembling-Hand Perfection Equilibrium

2.1.1 Dominant/Dominated Strategies

Definition (Best Response): A Strategy $\sigma_i \in \Sigma_i$ is a **Best Response** of player i for her rivals' strategies $\sigma_{-i} \in \Sigma_{-i}$ if

$$\forall \sigma_i' \in \Sigma_i,\ u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

Notice that *best response* depends on the rivals' strategies.

Definition (Strict Dominance): $\sigma_i \in \Sigma_i$ **strictly dominates** strategy $\sigma_i' \in \Sigma_i$ if

$$\forall \sigma_{-i} \in \Sigma_{-i},\ u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i})$$

Definition (Strictly Dominant Strategy): $\sigma_i \in \Sigma_i$ is a strictly dominant strategy of player i if σ_i strictly dominates all other strategies.

Remark: There is at most 1 strictly dominant strategy, and it need not exist. AND, if it exists, it must be a pure strategy and it is the unique *best response* to every strategy of rivals.

[8] So that we can focus on a player's strategic move based on their beliefs and not influenced by other players' actions like it would in a sequential game.

Definition (Strictly Dominated Strategy): $\sigma_i \in \Sigma_i$ is a strictly dominated strategy of player i if $\exists \sigma_i' \in \Sigma_i$ that strictly dominates σ_i.

Remark: If a *pure strategy* is a strictly dominated strategy, then so is any mixed strategy that plays it with non-zero probability[9].

Proposition (MWG 8.B.1): A pure strategy $s_i \in S_i$ is *strictly dominated* if and only if $\exists \sigma_i \in \Delta(S_i)$ such that

$$\forall s_{-i} \in S_{-i},\ u_i(\sigma_i, s_{-1}) > u_i(s_i, S_{-i})$$

Definition (Weak Dominance): $\sigma_i \in \Sigma_i$ is said to **weakly dominate** strategy $\sigma_i' \in \Sigma_i$ if

$$\forall \sigma_{-i} \in \Sigma_{-i},\ u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

Definition (Weakly Dominant Strategy): $\sigma_i \in \Sigma_i$ is a weakly dominant strategy of player i if σ_i weakly dominates all other strategies.

Definition (Weakly Dominated Strategy): $\sigma_i \in \Sigma_i$ is a weakly dominated strategy of player i if $\exists \sigma_i' \in \Sigma_i$ that weakly dominates σ_i.

Remark: Strict dominance implies weak dominance, so there is at most 1 weakly dominant strategy.

Example: Domination of a Pure Strategy by Mixed Strategies

Consider the following normal form game:

P_1 \ P_2	L	R
U	$\underline{10}$,1	0,4
M	4,2	4,3
D	0,5	$\underline{10}$,2

Notice that M is not dominated by U nor D. However, M is strictly dominated by the *mixed strategy* $\frac{1}{2}U + \frac{1}{2}D$

[9] Can be proven with independence axiom of rational choice

Example: Weakly Dominant and Dominated Strategies

Consider the following normal form game:

P_1 \ P_2	L	R
U	1,0	$\underline{-1}$, $\underline{1}$
D	$\underline{2}$,$\underline{5}$	$\underline{-1}$,3

Notice that

$$u_1(D_1, L_2) > u_1(U_1, R_2)$$
and
$$u_1(D_1, R_2) = u_2(U_1, R_2)$$

Since D_1 is the unique *best response* when P_2 plays L_2 and a best response when P_2 plays R_2, D_1 is a weakly dominant strategy.

2.1.2 Dominant Strategy Solutions (DSS)

Definition (Dominance Solvable): When every player has a dominant strategy, the game is said to have a dominant strategy solution. Assuming that all players should play their dominant strategy, we say that such games are **dominance solvable**.

Example: Dominance Solvable Game

Consider the following normal form game:

P_1 \ P_2	L	C	R
U	$\underline{2}$,1	3,$\underline{2}$	$\underline{1}$,-1
M	-1,5	4,$\underline{8}$	$\underline{1}$,7
D	$\underline{2}$,$\underline{1}$	$\underline{5}$,$\underline{1}$	$\underline{1}$,1

The dominant strategy solution in this game is (D, C) since D and C are both weakly dominant strategies.

2.1.3 Iterative Elimination of Strictly Dominated Strategies (IESDS)

The central question after studying dominance is: What if a player does not have a dominant strategy?

If a player is rational, then the player should, at least, not play any strictly dominated strategies. Since all players should assume that other players will not play a strictly dominated strategy, players do not have to consider playing *best responses* to other players' strictly dominated strategy. This means that we can iteratively reduce the size of the game and reduce the size of the set of potential solutions. Note that this means IESDS assumes as many levels of *common knowledge* as you need to iterate.

The general step-by-step process of IESDS is:

Step 1. Start from ANY player of the game and eliminate ALL *strictly dominated strategies* of that player[10]

Step 2. Look at the newly reduced game, and go to a different player and eliminate ALL *strictly dominated strategies* of that player

Step 3. Repeat until there is no strictly dominated strategies left for any player

Example: IESDS

Consider the following normal form game:

P_1 \ P_2	L	C	R
U	-2,2	-10,-1	2,-2
M	-1,-10	-5,-5	1,-7
D	-3,3	-11, 1	-1,2

We shall attempt to solve this game with IESDS:

Step 1: For player 1, D is strictly dominated by T and M.

[10] The order does not matter if you are eliminating strictly dominated strategies. However, the order does matter if you are eliminating weakly dominated strategies as well.

P_1 \ P_2	L	C	R
U	-2,2	-10,-1	2,-2
M	-1,-10	-5,-5	1,-7
~~D~~	~~-3,3~~	~~-11, 1~~	~~-1,2~~

Step 2: For player 2, in this once-reduced game, R is strictly dominated by C and L.

P_1 \ P_2	L	C	~~R~~
U	-2,2	-10,-1	~~2,-2~~
M	-1,-10	-5,-5	~~1,-7~~
D	-3,3	-11, 1	~~-1,2~~

Step 3: For player 1, in this twice-reduced game, T is strictly dominated by M.

P_1 \ P_2	L	C	R
~~U~~	~~-2,2~~	~~-10,-1~~	~~2,-2~~
M	-1,-10	-5,-5	1,-7
D	-3,3	-11, 1	-1,2

Step 4: For player 2, in this thrice-reduced game, L is strictly dominated by C.

P_1 \ P_2	~~L~~	C	R
U	~~-2,2~~	-10,-1	2,-2
M	~~-1,-10~~	-5,-5	1,-7
D	~~-3,3~~	-11, 1	-1,2

After IESDS, there is one strategy pair left and is the unique solution to this game

P_1 \ P_2	L	C	R
U	-2,2	-10,-1	2,-2
M	-1,-10	-5,-5	1,-7
D	-3,3	-11, 1	-1,2

We denote this surviving set of strategy profiles $IESDS(\Gamma_N) = \{(M, C)\}$.

2.1.4 Rationalizable Strategies

In certain situations, we may want something that is similar to IESDS but with more leeway so that some strategies that could be good in certain scenarios are still considered. The answer to that is *rationalizable strategies*.

Definition (Rationalizable Strategies): A **rationalizable strategy** is a strategy that survives **Iterative Elimination of Never Best Response Strategies.**[11]

Remark: $DSS \subseteq IESDS \subseteq RS/IENBRS$

Remark: A strategy is *NOT strictly dominated* if and only if it is a *best response* to some rivals' strategy.

2.2 Pure Strategy Nash Equilibrium

Fundamentally, the rationalizability of a strategy requires that it be a **best response** to some other reasonable strategy of the rivals. If we tighten that a little bit, we can look for the intersection of every player's *best response*. That is the concept of **Nash Equilibrium**.

Definition (Nash Equilibrium): A strategy profile $s = (s_1, \ldots, s_I) \in \prod_{i \in \mathcal{I}} S_i$ constitutes a **Pure-Strategy Nash Equilibrium** if $\forall i \in \mathcal{I}$, $s_i' \in S_i$,

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

The "tightening" that happens can be described as follows:

- Rationalizability: Requires that a player's strategy is a *best response* to some reasonable conjecture about their rivals' propensity to play their *best responses*.
- Nash Equilibrium: Adds that said conjecture must be correct.

[11] Recall that $s_i \in S_i$ is strictly dominated $\Rightarrow$ s_i is never a best response. The converse is only true in 2-player games.

Example:

Consider the following normal form game:

P_1 \ P_2	b_1	b_2	b_3	b_4
a_1	$0, \underline{7}$	$2, 5$	$7, 0$	$0, 1$
a_2	$5, 2$	$\underline{3}, \underline{3}$	$5, 2$	$0, 1$
a_3	$\underline{7}, 0$	$2, 5$	$0, \underline{7}$	$0, \underline{1}$
a_4	$0, 0$	$0, -2$	$0, 0$	$\underline{10}, -1$

First, see that b_4 is strictly dominated by the mixed strategy $\frac{1}{2}b_1 + \frac{1}{2}b_3$, so b_4 is never a best response.
Next, that once b_4 is eliminated, a_4 is strictly dominated by a_2, so a_2 is never a best response.
Notice that no further elimination can be done through *IENBRS* in the reduced game:

P_1 \ P_2	b_1	b_2	b_3	b_4
a_1	$0, \underline{7}$	$2, 5$	$7, 0$	$0, 1$
a_2	$5, 2$	$\underline{3}, \underline{3}$	$5, 2$	$0, 1$
a_3	$\underline{7}, 0$	$2, 5$	$0, \underline{7}$	$0, \underline{1}$
a_4	$0, 0$	$0, -2$	$0, 0$	$\underline{10}, -1$

Now, if we further require that the best responses have to correspond to each other, we are left with the strategy profile $(3, 3)$. This profile if the *Pure-Strategy Nash Equilibrium.*

Example: A game with a continuum of pure strategies

Two individuals consider undertaking a business venture that will earn them 100 dollars in profit, but they must agree on how to split the 100 dollars. Bargaining works as follows: The two individuals each make a demand simultaneously. If their demands sum to more than 100 dollars, then they fail to agree, and each gets nothing. If their demands sum to less than 100 dollars, they do the project, each gets his demand, and the rest goes to charity.

This game can be represented in normal form:

- $\mathcal{I} = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_i(s_1, s_2) = s_i \mathbb{1}\{s_1 + s_2 \leq 100\}$

Given that player 1 plays x, player 2's best response is to play $100 - x$. So the PSNE of this game is:

$$PSNE(\Gamma_N) = \{(s_1, s_2) \mid s_1 + s_2 = 100,\ s_1, s_2 \in [0, 100]\}$$

Proposition: In a non-finite strategic game, a PSNE always exists if:

(i) S_i is a non-empty, convex, and compact subset of $\mathbb{R}^n$

(ii) $u_i(s_1, \ldots, s_I)$ is continuous in $(s_1, \ldots, s_I)$ and quasi-concave in s_i

Proof: Existence of PSNE in Some Non-Finite Strategic Games

Lemma: Is $S_1, \ldots, S_I$ are non-empty and S_i is compact and complex, and $u_i(\cdot)$ is continuous in the strategy space and quasi-concave in S_i, then the **best response correspondence** $\beta_i(\cdot)$ is non-empty, convex-valued, and upper-hemi-continuous.

Sketch Proof: Since the strategy space is compact, the co-domain of the best response correspondence is also compact[a]. Since the domain and co-domain are compact and $u_i(\cdot)$ is continuous in the strategy space, we know that $\beta_i(\cdot)$ is non-empty[b] and upper-hemi-continuous[c]. Since S_i is convex-valued and $u_i(\cdot)$ is continuous, $\beta_i(\cdot)$ is also convex-valued.

Using this Lemma, $\beta_i(\cdot)$ is a *non-empty valued, convex-valued, and upper-hemi-continuous* self-mapping that is on a compact, convex, and non-empty space. By *Kakutani's fixed point theorem*, there exists at least 1 fixed point $s_i \in S_i$ such that $s_i \in beta_i(s_i)$.

If such fixed point exists for all players in the game (i.e., all S_i's satisfy the conditions, then the tuple of all these fixed points constructs the Pure Strategy Nash Equilibrium.

□

[a] See proof of *Weistrass Theorem of Maximum* for more details.
[b] By *Weistrass Theorem of Maximum.*
[c] By *Berge's Theorem of Maximum*

2.3 Mixed Strategy Nash Equilibrium

Definition (MSNE): A strategy profile $\sigma^* \in \prod_{\iota \in \mathcal{I}} \Sigma_i$ is a **Mixed Strategy Nash Equilibrium** if $\forall i \in \mathcal{I}$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*),\ \forall \sigma_i' \in \Sigma_i$$

Solution Intuition: An MSNE to player i is a mixed strategy that makes player $-i$'s (her rivals) indifferent between their own pure strategies by creating "certainty" from uncertainty.

Example: Mixed Strategy Nash Equilibrium

Consider the Simultaneous Matching Pennies
$\Gamma_N = (\mathcal{I}, (S_1, S_2), (u_1, u_2))$
$\mathcal{I} = \{1, 2\}$, $S_1 = \{H_1, T_1\}$, $S_2 = \{H_2, T_2\}$

P_1 \ P_2	H_2	T_2
H_1	$-1, \underline{1}$	$\underline{1}, -1$
T_1	$\underline{1}, -1$	$-1, \underline{1}$

It should be clear by now that this game has no PSNEs due to the uncertainty. Hence, the players can reach an MSNE by creating certainty about the uncertainty.
Consider the mixed strategy

$$\sigma_1(H_1, T_1) = (p, 1-p)$$
$$\sigma_2(H_2, T_2) = (q, 1-q)$$

We know that if P_1 plays one side with a higher probability than the other, P_2 will deviate to playing the opposite side for certainty, in order to maximize their expected payoffs. As such, P_1 needs to mix their strategy to the point where P_2 must be indifferent between playing one or the other with certainty.

$$u_2(H_2, \sigma_1) = p \cdot 1 + (1-p) \cdot (-1) = p \cdot (-1) + (1-p) \cdot 1 = u_2(T_2, \sigma_1)$$
$$p = \frac{1}{2}$$

Similarly, P_2 will want to mix her strategy so that P_1 is indifferent between playing either side with certainty.

$$u_1(T_1, \sigma_2) = q \cdot 1 + (1-q) \cdot (-1) = q \cdot (-1) + (1-q) \cdot 1 = u_1(H_1, \sigma_2)$$
$$p = \frac{1}{2}$$

So the unique MSNE in this game is:

$$\sigma_{MSNE} = \left(\sigma_1\left(H_1, T_1\right) = \left(\frac{1}{2}, \frac{1}{2}\right), \sigma_2\left(H_2, T_2\right) = \left(\frac{1}{2}, \frac{1}{2}\right)\right)$$

Proposition: An **MSNE** always exists in finite-strategic games[12].

2.4 Trembling Hand Perfect Nash Equilibrium

Recall that even though IESDS gives us the same surviving strategies regardless of the order of elimination, that is not the case for IEWDS (and generally, IEWDS $\subset$ IESDS). This nuance pushes us to ask, what happens if a player "accidentally" played a weakly **dominated** strategy? Should such "equilibrium" be sustained as a refinement of Nash equilibrium?

Motivating a Refinement of Nash Equilibrium Consider the following game
$\Gamma_N = (\mathcal{I}, (S_1, S_2), (u_1, u_2))$
$\mathcal{I} = \{1, 2\}$, $S_1 = \{U, D\}$, $S_2 = \{L, R\}$

P_1 \ P_2	L	R
U	$1, 1$	$0, 0$
D	$0, 0$	$0, 0$

There are 2 PSNEs in this game, (U, L) and (D, R). But we can clearly tell that (D, R) is not a good outcome, and P_2 should definitely play L with certainty if they think P_1 would play U with non-zero probability.

How can we put a different qualifier on (U, L) such that we don't think of these two PSNEs as equally "good" solutions?

The answer is - Trembling Hand Perfect Nash Equilibrium

[12] This means a game with finitely many pure strategies.

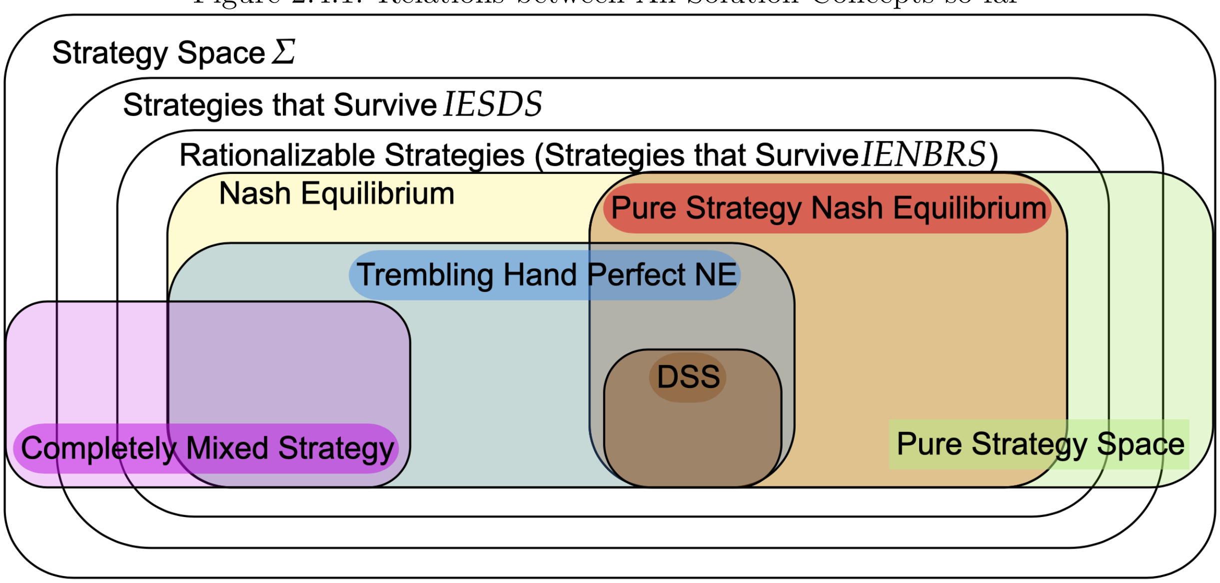
Definition (Perturbation): A game $\gamma_\varepsilon = \{\mathcal{I}, (\Delta_\varepsilon(S), U)\}$ is a **perturbed game** with perturbation ε where every player i plays every non-strictly dominated pure strategy with at least some non-zero but small probability $\varepsilon(s_i)$.

Definition (THPNE): A strategy profile $\sigma \in \prod_{i\in\mathcal{I}} \Sigma_i$ is a **Trembling Hand Perfect Nash Equilibrium** if there exists a sequence of perturbed games $\Gamma_{\varepsilon^k} \to \Gamma$ for which there exists a sequence of MSNE $\sigma^k \to \sigma$.

Proposition: Every finite strategic game has at least one THPNE.

Proposition: A strategy profile $\sigma \in \prod_{i\in\mathcal{I}} \Sigma_i$ is a **Trembling Hand Perfect Nash Equilibrium** if and only if there exists a sequence of completely[13] mixed strategy profiles $\sigma^k \to \sigma$ such that σ_i is a *best response* to σ^k_{-i} for all i and all k.

Figure 2.4.1: Relations between All Solution Concepts so far



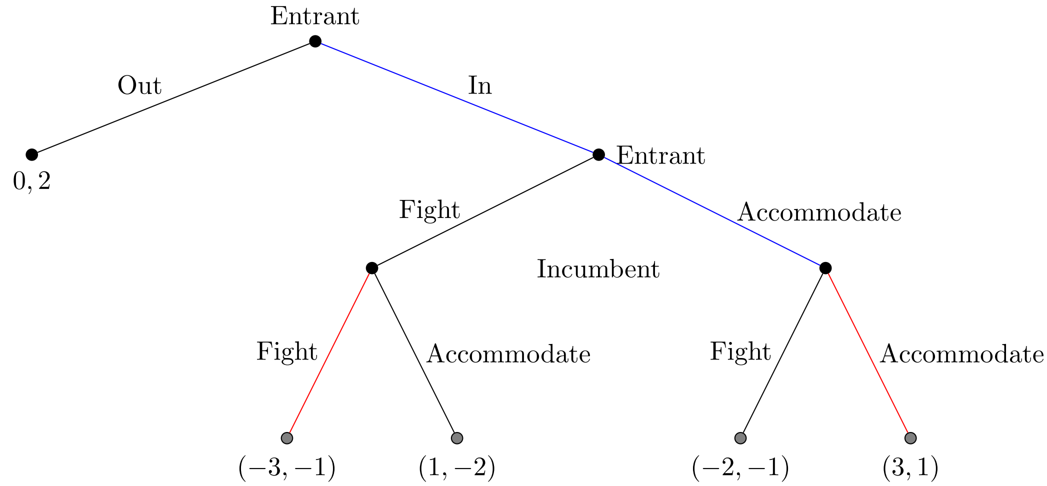
2.5 Subgame Perfect Nash Equilibrium

So far, we have been focusing our discussion of solution concepts on simultaneous games. Now that we have those tools, it is time to look at Dynamic/Sequential games.
The principle of such studies should be fairly intuitive - a player's strategy in a dynamic game should specify optimal actions at every information set of the game.

[13]This implies that all completely mixed strategy Nash equilibria are trivially trembling hand perfect.

Consider the following extensive game. If we want to try to predict the outcome of the game, we need to specify what the players would do at each of the information sets/nodes so that their actions are optimal.

Figure 2.5.1: Basic Entrant/Incumbent Game



Intuitively, the easiest way to do so, is to start from the terminal nodes, and work our way back up[14]. It should be obvious that the PSNE in this game is $((In, Accommodate), (Accommodate))$. Since both player can observe each others' actions, in this game, they would easily pick the most mutually beneficial strategies. Even if the incumbent threatens to fight regardless, the Entrant know that the incumbent's threat is not credible and hence it would not happen. The way we worked back from the terminal nodes is called **backward induction**.

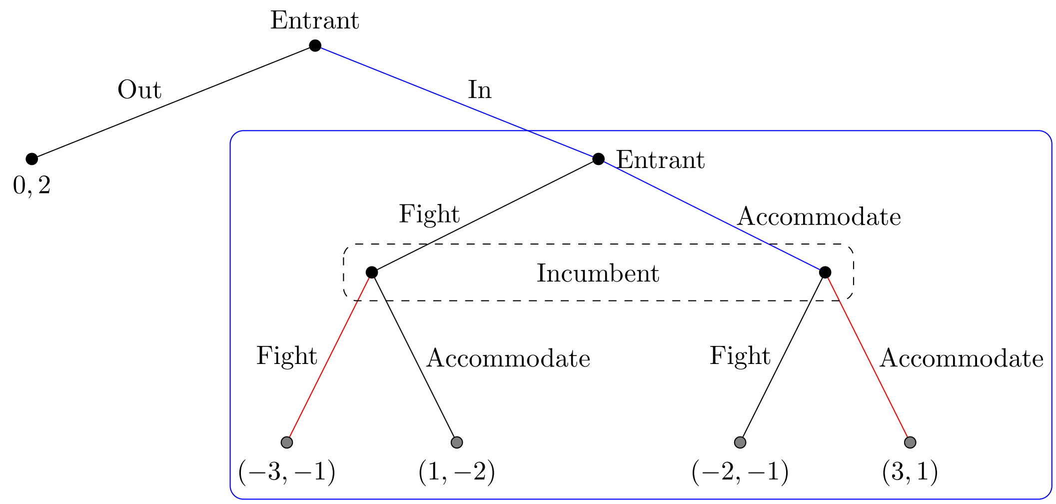
Proposition (Zermelo's Theorem): Every finite strategic game with perfect information has a PSNE that can be derived from backward induction. Moreover, if there is no tie in the payoffs between terminal nodes, the PSNE is unique.

But what if the game does not have perfect information? Does backward induction still work? Is it possible that we can create "certainty" from uncertainty like in the case of Mixed Strategy Nash Equilibrium?

[14]In this example, red represents P_1's optimal action, and blue represents P_2's optimal actions.

Consider a revised game from Figure 2.5.1 where the incumbent cannot observe the entrant's actions:

Figure 2.5.2: Basic Entrant/Incumbent Game with Imperfect Information



The entire game can be represented in normal form as:

Entrant \ Incumbent	Fight	Accommodate
In, Fight	-3,$\underline{-1}$	1,-2
In, Accommodate	$\underline{-2}$,1	$\underline{3}$,$\underline{1}$
Out, Fight	$\underline{0}$,$\underline{2}$	0,$\underline{2}$
Out, Accommodate	$\underline{0}$,$\underline{2}$	0,$\underline{2}$

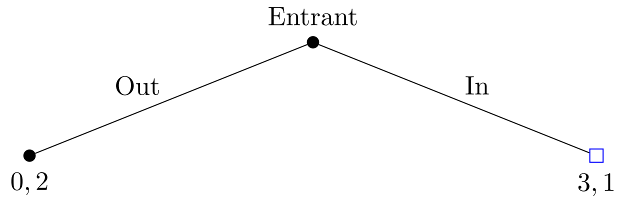
So there are 3 PSNEs in this game:
$((In, Accomodate), (Accommodate)), ((Out, Accomodate), (Fight)), ((Out, Fight), (Fight))$.
But are these equally "good"?

Notice that, within the blue box, we essentially have a simultaneous game, so we can represent this game in normal form as:

E \ I	Fight	Accommodate
Fight	-3,$\underline{-1}$	1,-2
Accommodate	$\underline{-2}$,1	$\underline{3}$,$\underline{1}$

Since $(Accomodate, Accommodate)$ is the unique PSNE in this game, we can actually reduce the blue box into just one node, as we know what the result of the blue box game would be. Meaning that the new game is:

Figure 2.5.3: Reduced Entrant/Incumbent Game



So between *In* and *Out*, the Entrant should choose *In*, meaning that, somehow, *((In,Accommodate),(Accommodate))* is a "better" NE than the other two PSNEs. How do we explain this result? What are the "rules" we can use to refine NEs?

Definition (Subgame): A **subgame** is a game that satisfy the following properties:

(i) It begins at a singleton information set and contains all successor nodes of the node

(ii) No information set is broken. If a node is in this game, then all other nodes in the same information set must also be in this game

The blue box from Figure 2.5.2 is a proper subgame[15] of the game.

Definition (SPNE): A strategy profile $\sigma \in \prod_{i \in \mathcal{I}} \Sigma_i$ is a **Subgame Perfect Nash Equilibrium** if it is a Nash Equilibrium in every subgame.

Proposition: Every finite strategic game of imperfect information has at least one SPNE. Moreover, if there is no tie in the payoffs between terminal nodes, the SPNE is unique.
Like in our motivating example, we can solve for SPNEs in a game through backward induction. The general step-by-step process is:

Step 1. Start from the end of the game tree and identify all the Nash Equilibrium (both PSNEs and MSNEs) in each subgame that does not contain another subgame.

Step 2. Use the found NEs' payoffs to reduce the subgames down to just a node (as in figure 2.5.3)

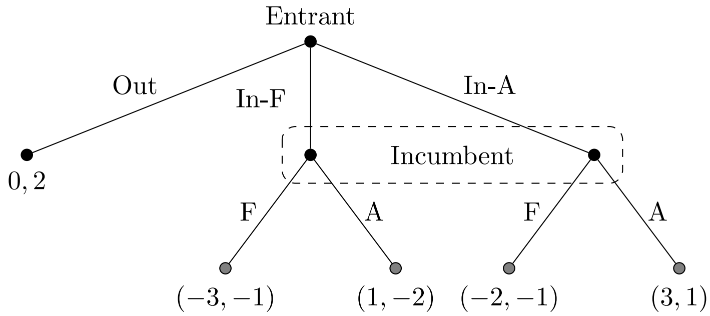
Step 3. Repeat step 1 and 2 until there is no proper subgames left

Note: If none of the subgames has more than 1 NE, then the SPNE found is unique. If there are multiple NEs in any subgames, do step 2 for each of the NE's payoff.

[15]The term *proper subgame* refers to any subgame that is not the main game itself.

Now, consider a slightly revised version of Figure 2.5.2 where the Entrant has 3 startegies: $Out, In - A, In - F$.

Figure 2.5.4: Basic Entrant/Incumbent Game with Imperfect Information



Notice that this game is essentially the same as that of 2.5.2, but there is no longer any proper subgame. In this case, there are 2 NEs, and hence 2 SPNEs: (Out, F) and $(In - A, A)$. The former is an NE because the incumbent knows that as long as they commit to fighting, than the entrant would stay out. But is such threat actually credible?

To answer that, we need some new tools about what the players believe would happen, taking us to the next solution concept - **Weak Perfect Bayesian Equilibrium**.

2.6 Perfect Bayesian Equilibrium

2.6.1 Sequential Rationality

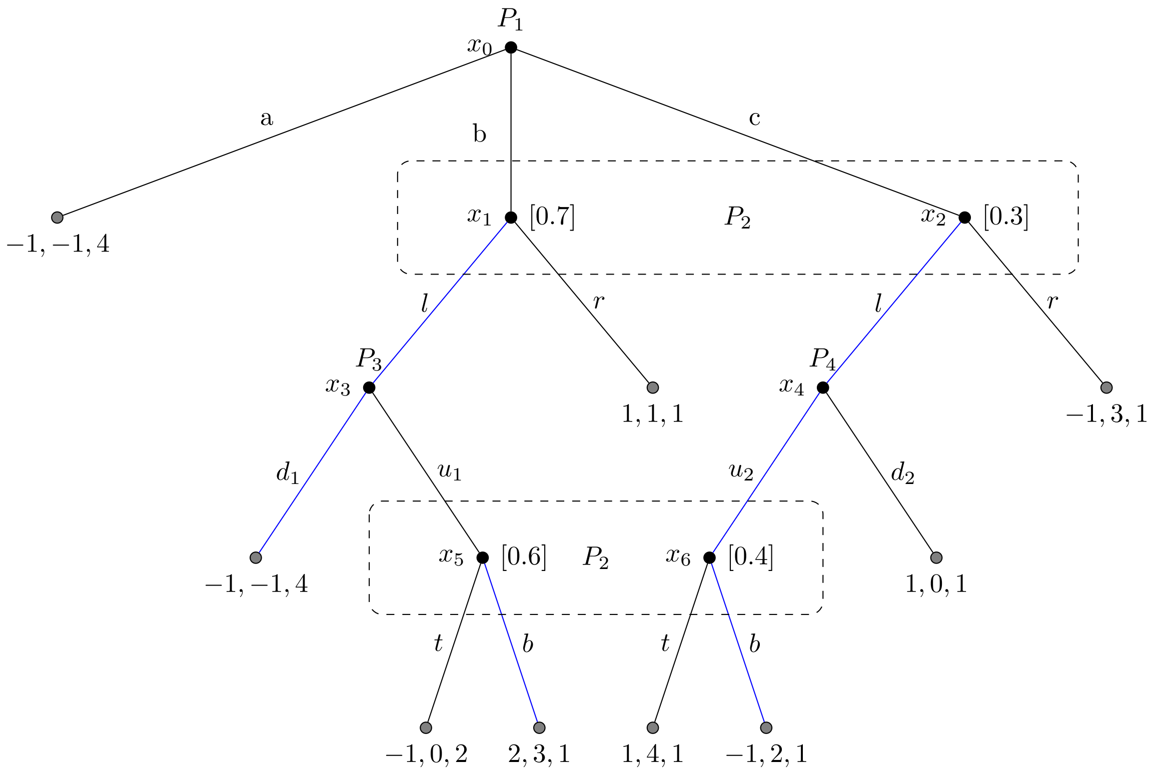
Definition (System of Beliefs): A **system of beliefs** μ is a specification of a probability distribution for each node x such that $\forall H \in \mathcal{H}, \sum\limits_{x \in H} \mu(x) = 1$.

Definition (SR): A strategy profile $\sigma \in \prod\limits_{i \in \mathcal{I}} \Sigma_i$ is said to be **Sequentially Rational** given the belief system μ if $\forall H \in \mathcal{H}$,

$$E[u_{\iota(H)} \mid H, \mu(H), \sigma_{\iota(H)}, \sigma_{-\iota(H)}] \geq E[u_{\iota(H)} \mid H, \mu(H), \sigma'_{\iota(H)}, \sigma_{-\iota(H)}], \forall \sigma'_{\iota(H)} \in \Sigma_{\iota(H)}$$

Example: Calculating Expected Payoffs Given A System of Beliefs

Consider the following game and the strategy profile σ where $\sigma_1(a,b,c) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $\sigma_2(lb) = 1$ and $\sigma_3(d_1 u_2) = 1$.



Player 2's expected payoff at information set $\{x_5, x_6\}$ given μ is:

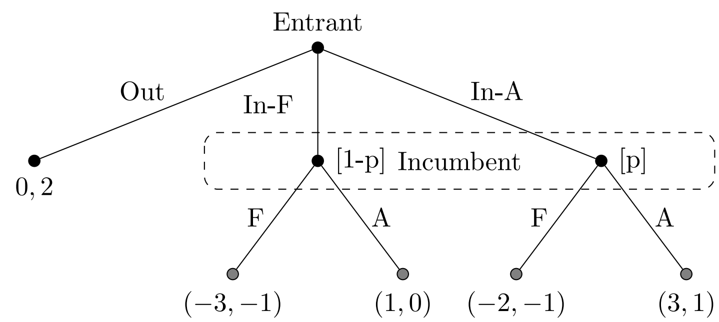
$$E[u_2 \mid \{x_5, x_6\}, \mu, \sigma] = 0.6 \cdot (3) + 0.4 \cdot (2) = 2.6$$

Player 2's expected payoff at information set $\{x_1, x_2\}$ given μ is:

$$E[u_2 \mid \{x_1, x_2\}, \mu, \sigma] = 0.7 \cdot (-1) + 0.3 \cdot (2) = -0.1$$

Looking back to our Entrant/Incumbent game in Figure 2.5.4 but now with a system of belief and, for simplicity, slightly different payoffs, can we find something new?

Example: A "Stupid" Belief



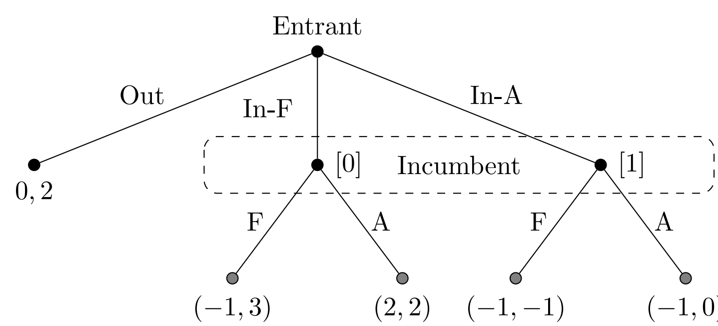
Claim: No strategy profile involving F is sequentially rational given any system of beliefs.

Proof: The incumbent's expected payoffs are:

$$E[u_I \mid H, A, \sigma_I] = (1) \cdot p + 0 \cdot (1-p) = p$$
$$E[u_I \mid H, F, \sigma_I] = (-1) \cdot p + (-1) \cdot (1-p) = -1 < p$$

Since F is strictly dominated by A, F and any strategies involving F cannot be sequentially rational.

Now, let's change the payoffs a little bit more



In this game, the strategy profile $(In - F, A)$ is sequentially rational given the belief, since the incumbent believed that the entrant chose $In - A$. However, this strategy profile is not a Nash Equilibrium, since A is not a best response to $In - F$. So the belief that "The incumbent plays $In - A$ with probability 1" is stupid.

This example shows us that it is not just important that a strategy profile is NE given the beliefs. The beliefs being not stupid is also important. This requirement of logical beliefs is called **consistency**.

2.6.2 Consistency in Beliefs

Definition (WPBE): The tuple (σ, μ) of a strategy profile $\sigma \in \prod_{i \in \mathcal{I}} \Sigma_i$ and a belief system μ is a **Weak Perfect Bayesian Equilibrium** if

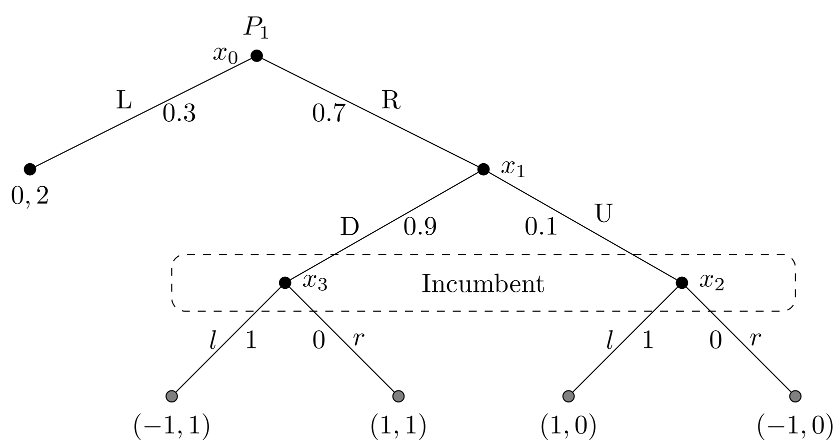
(i) σ is *sequentially rational* given μ

(ii) (Consistency) μ is derived from σ using Bayes' rule wherever possible

$$\mu(x) = P(x|H, \sigma) = \frac{P(x \mid \sigma)}{P(H \mid \sigma)}$$

Comparing the two examples from the previous page to the example in Figure 2.5.4, we can see that the difference between a NE and a WPBE is: NEs only need to be *sequentially rational* given μ at all information sets H such that $P(H \mid \sigma) > 0$. In other words, NEs only care about beliefs on the equilibrium path, which is why in Figure 2.5.4, (Out, F) is a NE. This means that $WPBE \subseteq NE$.

Example: Deriving a Consistent Belief



Here, a consistent μ at x_2 would have

$$\mu(x_2) = \frac{P(x_2 \mid \sigma)}{P(\{x_2, x_3\} \mid \sigma)} = \frac{0.7 \cdot 0.1}{0.7 \cdot 0.9 + 0.7 \cdot 0.1} = 0.1$$

Definition (PBE): The tuple (σ, μ) of a strategy profile $\sigma \in \prod_{i \in \mathcal{I}} \Sigma_i$ and a belief system μ is a **Perfect Bayesian Equilibrium** if it is a WPBE in every subgame (like how an SPNE is a NE in every subgame).

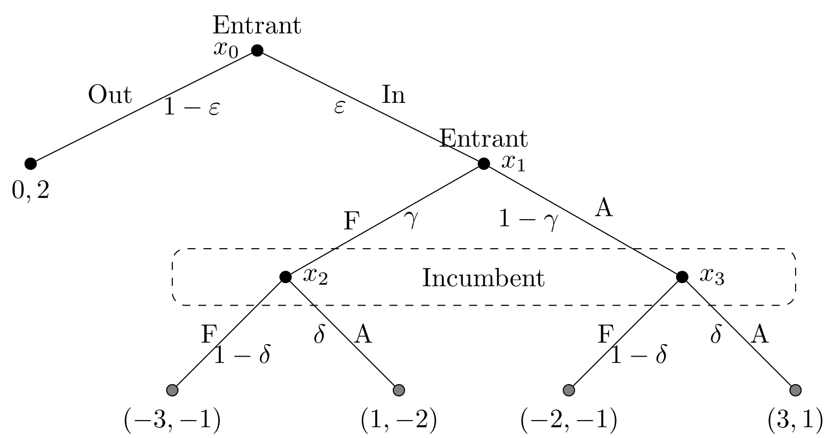
Definition (SE): The tuple (σ, μ) of a strategy profile $\sigma \in \prod_{i \in \mathcal{I}} \Sigma_i$ and a belief system μ is a **Sequential Equilibrium** if

(i) σ is *sequentially rational* given μ

(ii) There exists a sequence $(\sigma^k, \mu^k) \to (\sigma, \mu)$ such that μ^k's are derived from σ^k using Bayes' rule wherever possible.

Note: An SE is to PBE and WPBE NOT like a THPNE is to SPNE and NE.

Example: Sequential Equilibrium



Claim: $((In, A), A)$ with $\mu(x_3) = 1$ is a sequential equilibrium.

Proof:

$$u_I(A|x_3) = 1 > -1 = u_I(F|x_3)$$
$$u_I(A|x_2) = -2 < -1 = u_I(F|x_2)$$

So playing $\delta \in (0, 1)$ is sequentially rational if and only if

$$\mu(x_3) \cdot 1 - 2 \cdot (1 - \mu(x_3)) = -1 \;\Rightarrow\; \mu(x_3) = \frac{1}{3}$$

and playing $\delta = 1$ is sequentially rational if and only if

$$\mu(x_3) > \frac{1}{3}$$

Take the sequence $(\varepsilon^k, \gamma^k) \to (1, 0)$ so $\mu^k(x_3)$ is

$$\mu^k(x_3) = \frac{P(x_3 \mid \sigma^k)}{P(\{x_2, x_3\} \mid \sigma^k)} = \frac{\varepsilon^k \cdot (1 - \gamma^k)}{\varepsilon^k(1 - \gamma^k) + \varepsilon^k \gamma^k} = 1 - \gamma^k \to 1$$

So $\mu(x_3) = 1$ is consistent, and $((In, A), A)$ is SR givent $\mu(x_3) = 1$, so $((In, A), A)$ with $\mu(x_3) = 1$ is a sequential equilibrium.

Claim: $((Out, A), F)$ with any μ is NOT a sequential equilibrium.

Proof:

Suppose otherwise, that this strategy profile can form a sequential equilibrium. We need a sequence $(\varepsilon^k, \gamma^k, \delta^k) \to (0, 0, 0)$. But as shown in the proof above, for $\gamma^k < \frac{2}{3}$, we have $\mu^k(x_3) > \frac{1}{3}$, leading to $\delta \to 1$.
In fact, for any sequentially rational strategy with $\delta \to 0$, we must have that $\mu^k(x_3) \to \mu(x_3) \in [0, \frac{1}{3})$.
But such belief will be inconsistent with the strategy $1 - \gamma^k \to 1$.
Hence the strategy profile $((Out, A), F)$ cannot be a sequential equilibrium with any belief that is consistent given such strategy.

3 Applications in the Market Environment

3.1 Solutions in Static Models

Remember how we talked about using game theory to study the effect of information asymmetry on market inefficiencies? This is the time!

We will begin by discussing monopolistic pricing, where one producer dictates the quantity supplied, and hence prices, of the goods market.

3.1.1 Monopolistic Pricing

In the monopolistic producer market environment, we have:

- One single producer in the good market
- The demand for this good given by the **market demand function** $Q(p)$ such that
  - $Q(p)$ is continuous, strictly decreasing, and strictly positive
  - $\exists \bar{p} < \infty$ such that $\forall p \geq \bar{p}$, $Q(p) = 0$
  - The **inverse market demand function** is $P(q) \equiv Q^{-1}(q) = \min\{p \mid Q(p) = q\}$
- The cost of producing q units is given by $c(q)$
  - $P(\cdot)$ and $c(\cdot)$ are C^2 functions
  - $P(0) > c'(0)$, $c'' > 0$
  - $\exists!\, q^0$ such that $P(q^0) = c'(q^0)$

The monopoly profit maximization problem is:

$$\max_q \ \pi(q) = q \cdot P(q) - c(q)$$

which gives us the F.O.C.:

$$P(q) + \underbrace{P'(q) \cdot q}_{\substack{\text{Market externality} \\ \text{of price change}}} = c'(q) \ \Rightarrow \ P(q) = c'(q) - \underbrace{P'(q)}_{<0} q$$

Notice that different from the competitive equilibrium producer first order condition, the monopolisitc one has an extra term of $P'(q) \cdot q$. This is because in the infinitely many producer environment, a firm's change in production quantity does not affect the price of the good. However, in a monopolistic producer environment, the changes in quantity supplied directly affect the price of the good. As a monopoly, the producer can transfer all that externality to consumers by pricing the good higher than the marginal cost by $|P'(q) \cdot q|$.

Figure 2.1.1 illustrates this inefficiency: q^m is the monopolistic equilibrium quantity and q^{CE} is the competitive equilibrium quantity. MR is the marginal revenue curve, $c'(q)$ is the marginal cost curve.

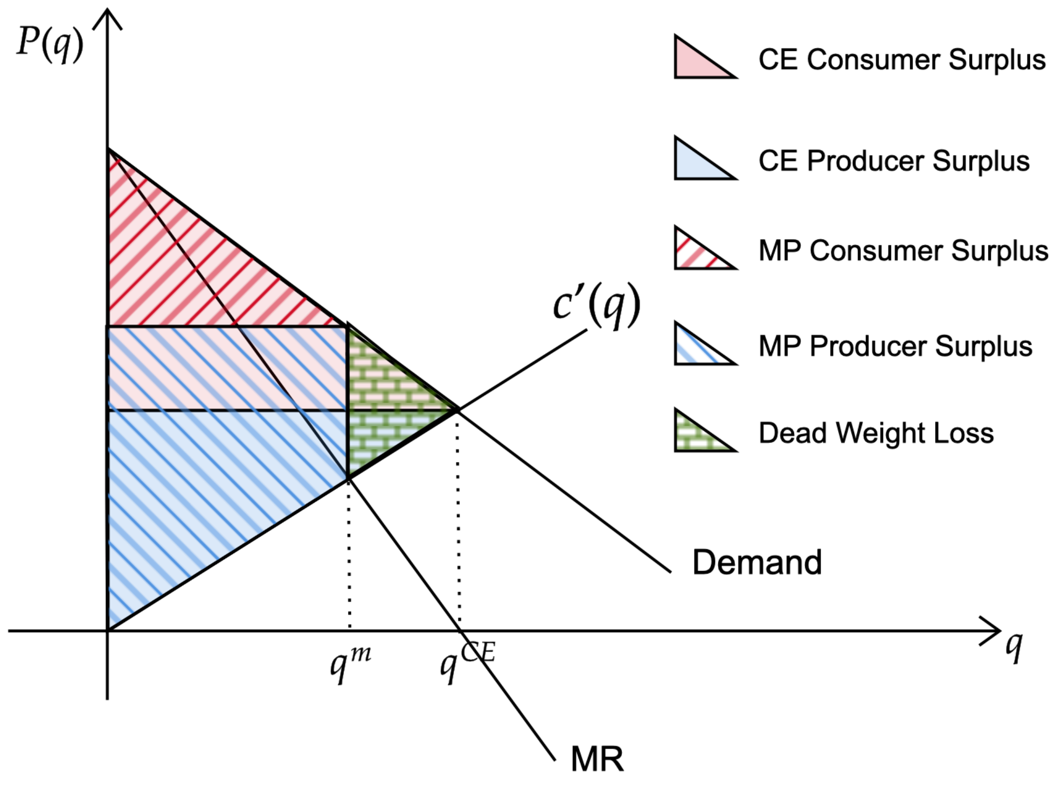


Figure 3.1.1: Monopolistic Pricing Causing Dead Weight Loss

3.1.2 Bertrand Model of Price Competition

In the Bertrand market environment, we have:

- 2 producers simultaneously choose prices p_i and p_j
- Both firms produce the same good at the same constant marginal cost c
- The market demand function is $Q(p)$

- Buyers only buy from the firm that charges the lower price, so the supply function for firm i is:

$$Q_i(p_i, p_j) = \begin{cases} 0 & , p_i > p_j \\ \frac{1}{2}Q(p_i) & , p_i = p_j \\ Q(p_i) & p_i < p_j \end{cases}$$

- Firm i's profit is $\pi_i(p_i, p_j) = (p_i - c) \cdot Q_i(p_i, p_j)$

Claim: There exists an unique Nash Equilibrium at $p_i^* = p_j^* = c$

Proof:

Step 1: Verify that $p_i^* = p_j^* = c$ is indeed a Nash Equilibrium
At $p_i^* = p_j^* = c$, WLOG, if firm i deviates to $p_i - \varepsilon$, $\varepsilon > 0$, then their profit is $(p_i - \varepsilon - c)Q_i(p_i - \varepsilon) < 0$, so firm i would not want to cut price below c. If firm i deviates to $p_i + \varepsilon$, $\varepsilon > 0$, then all consumers would buy from firm j, so firm i would not deviate to raise price above c.
By symmetry, $p_i^* = p_j^* = c$ is a Nash Equilibrium.

Step 2: Verify that the Nash Equilibrium is unique
Given that if the two firms prices are different, one firm would have 0 sales, any NE must satisfy $p_i^* = p_j^*$.
Suppose that $p_i^* = p_j^* < c$, then both firms would either deviate towards c because they would be making negative profits otherwise.
Suppose that $p_i^* = p_j^* > c$, then both firms would deviate towards c because pricing strictly less than and sufficiently close to the other firm's price strictly dominates not doing so.
Hence the NE at $p_i^* = p_j^* = c$ is unique.

Remark: The uniqueness of the NE only holds if there are 2 firms. When there are more than 2 firms, we would have a continuum of NEs because firms can collude, in which case pricing strategy of 1 firm may not always affect sales. □

3.1.3 Cournot Model of Quantity Competition

In the Cournot market environment, we have:

- 2 producers simultaneously choose quantities q_i and q_j
- Both firms produce the same good at the same constant marginal cost c
- The total quantity produced is $Q = q_i + q_j$
- The inverse demand is $P(Q)$ such that P is differentiable, strictly decreasing, and $P(0) > c$
- Firm i's profit is $\pi_i(p_i, p_j) = [P(q_i + q_j) - c] \cdot q_i$

In this duopoly setup, firm i's first order condition is:

$$P(q_i + q_j) + P'(q_i + q_j)q_i = c$$

In Nash Equilibrium, we have $q_i^d = q_j^d = q^d$. In fact, this NE applies to all $n-$firms.

3.2 Solutions in Dynamic Models

3.2.1 Stackelberg Model

In the Stackelberg market environment, we have:

- 2 producers (leader and follower) sequentially choose quantity q_1 and q_2
- Both firms produce the same good at the same constant marginal cost c
- Firm i's profit function is $\pi_i(q_1, q_2) = p(q_1 + q_2) \cdot q_i - c \cdot q_i$

Proposition: The unique PSNE is $(q_1^s, BR_2(q_1^s))$ where

$$q_1^s \equiv \underset{q_1 \geq 0}{argmax}\ \pi_1(q_1, BR_2(q_1))BR_2(q_1^s) \equiv \underset{q_1 \geq 0}{argmax}\ \pi_1(q_1, BR_2(q_1))$$

Example: Stackelberg Model with Linear Demand

Suppose that firm i's profit function is:

$$\pi_i(q_1, q_2) = [a - b(q_1 + q_2) - c] \cdot q_i$$

Using backward induction, the F.O.C of the Stackelberg follower's interior solution is:

$$a - bq_1 - 2bq_2 - c = 0 \ \Rightarrow \ q_2^s = \frac{a - c}{2b} - \frac{1}{2}q_1^s$$

The Stackelberg leader's problem is:

$$\max_{q_1} \left\{0, \left[a - b(\left(\frac{a-c}{2b} - \frac{1}{2}q_1 + q_1\right) - c\right] \cdot q_1\right\}$$

So the F.O.C. of the Stackelberg leader's interior solution is:

$$a - \frac{a-c}{2} - bq_1 - c = 0 \;\Rightarrow\; q_1^s = \frac{a-c}{2b}$$

By assumption $a > c$, so the leader produces a non-zero quantity. Subbing in q_1^s we get

$$q_2^s = \frac{a-c}{2b} - \frac{1}{2}q_1^s = \frac{a-c}{2b} - \frac{a-c}{4b} = \frac{a-c}{4b}$$

Recall that, under the Cournot Duolpoly quantity competition model, the equilibrium out come is $q_1^d = q_2^d = \frac{a-c}{3b}$. How exactly did the dynamic model change this result?

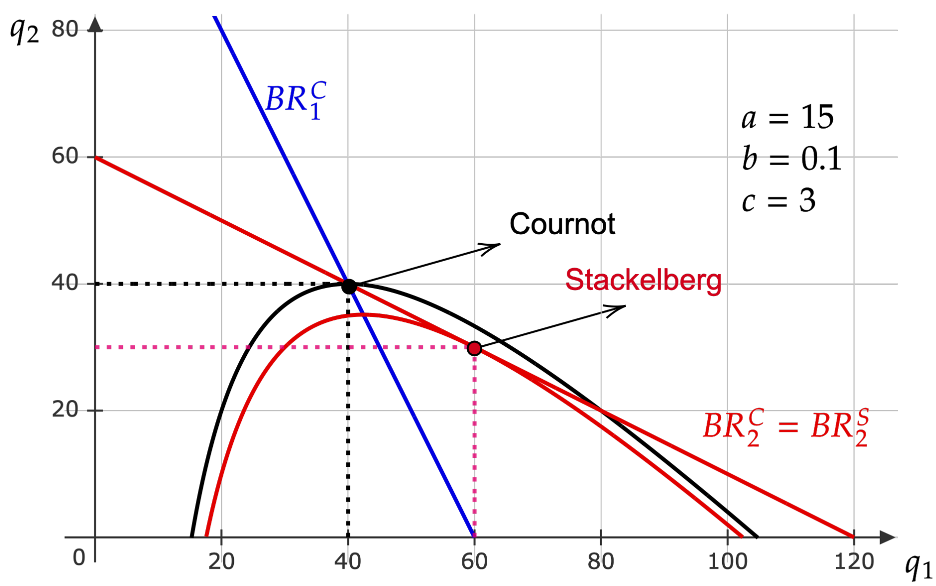
The following graph illustrates this idea. Essentially, in the Cournot model, since both firms are choosing simultaneously, they have to match each other's best responses. And any deviation would not be profit-maximizing.

However, in the Stackelberg model, the leader knows that the follower has to respond to whatever the leader decides. This means that the leader can commit to a higher quantity and thus force the follower to produce at a lower quantity in equilibrium.

Graphically, this means that the iso-profit curve for the leader has slope 0 at the Cournot quantity (because firm 2 max q_2 holding π_1 constant).

But the iso-profit curve for the leader will have the same slope as their best response line at the Stackelberg quantity (because the leader gets to make sure that q_2 responds to q_1 in the sequential game).

Figure 3.2.1: Stackelberg vs. Cournot



4 Repeated Games

So far, we have studied models and solution concepts that are based on the logic that the game ends after all the players moved, and there is often no recourse for bad faith actions. However, that is not how life works. In any circumstances, I can do things that are completely selfish and out of whack, and I totally have free will to do so. I CHOOSE not to do so because I would have to face the consequences of my bad actions.

This is how the study of repeated games got started. In essence, we want to figure out what happens if players have to face other players again in the same game structure. Would they rationally choose bad faith actions? Or would they cooperate to avoid being screwed over in the next round?

Just like optimization problem, we need to study this in the short-run (finite horizon) and long-run (infinite horizon.

I hope you are as excited as I am. Let's get started!

4.1 Finitely Repeated Games

Consider the following prisoner's dilemma game:

P_1 \ P_2	C	D
C	$\underline{2}, \underline{2}$	$\underline{6}, 1$
D	$1, \underline{6}$	$5, 5$

It should be easy for readers to see that even though (C, C) is our classic Nash equilibrium, (D, D) would be a Pareto improvement. But is it possible for this to happen? Is there a way to incentivize players to not deviate?

In the spirit of our motivation at the beginning of this section, let's see that happens if this game is played twice consecutively:

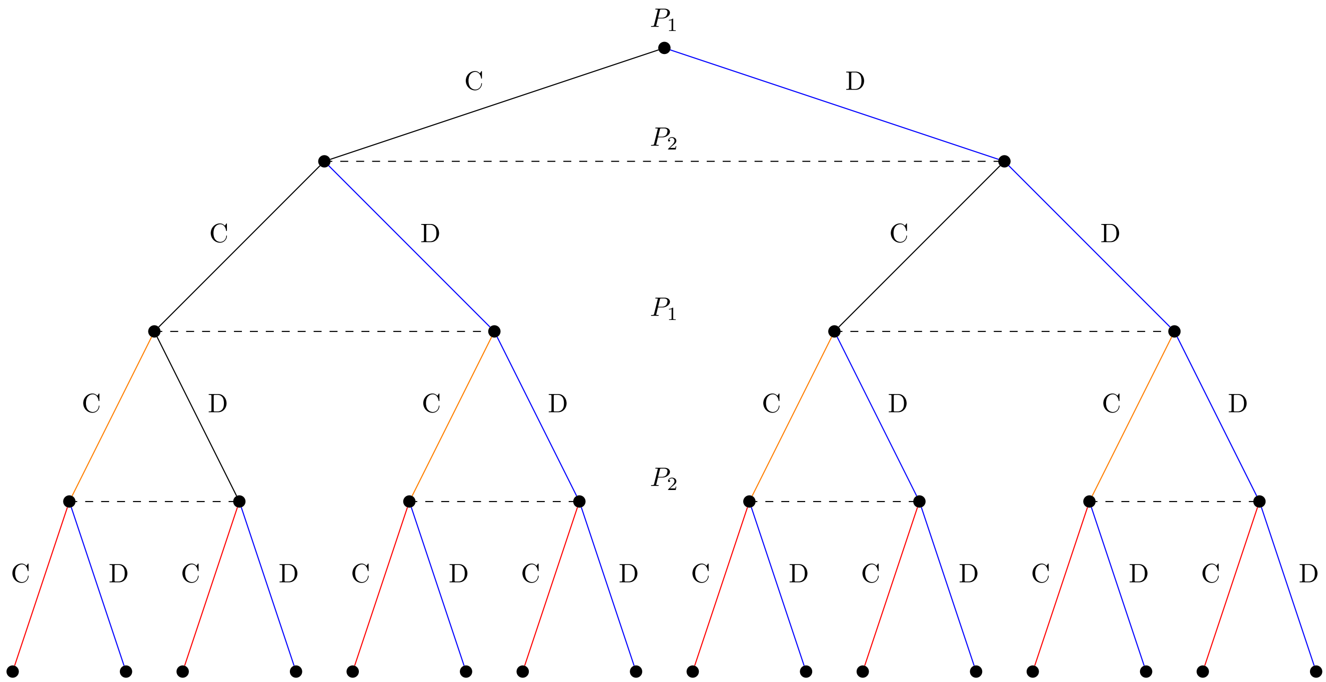
Consider the blue strategy profile where both players play the Pareto optimal. Does P_2 have an incentive to deviate to C in the second game?
The answer is yes, and it should be intuitive to think that because (C, C) is the **unique NE**

in the non-repeated version of the game.

Knowing this, P_1 then has an incentive to deviate to C in the second game as well.

You can probably what happens next. In the end, the entire game will revert back to playing (C, C) in every repetition, even for a large number of repetitions.

Figure 4.1.1: Finitely Repeated Prisoner's Dilemma



Does this means all hopes are lost? Not quite. Notice that the reason that cooperation did not work out is that in the single non-repeated game, there is an unique Nash Equilibrium. What that means is that even if the game is repeated finitely many times, the only SPNE is to play the NE EVERY SINGLE TIME. So what did we learn from this?

Definition: A **Stage Game** is a complete game that gets repeated in a repeated game.

Proposition: If a *stage game* has an unique Nash Equilibrium, then the finitely repeated game of such stage game has an unique Subgame Perfect Nash Equilibrium where the NE is played in every stage.

Definition: A **finitely repeated game** is a stage game $\Gamma_N = \{\mathcal{I}, \{S_i\}, \{u_i\}\}$ that is repeated $T \in \mathbb{N}$ times.

What if the stage game has multiple equilibria? Does that make finitely repeated games more interesting. The answer is "Yes! Kind of...not really :("

Proposition: If the stage game has multiple Nash Equilibria, then there exists *Subgame Perfect Nash Equilibria* in the finitely repeated game with non-NE cooperation where the SPNE follows:

- Reward non-NE cooperation by playing the "good" NE at a later stage
- Punish deviation from cooperation by playing the "bad" NE at a later stage

Definition: The t'th **History** of the game, denoted H^t, is the realized strategy profile in stage t.

Example: Finitely Repeated Games with Multiple NEs

Consider the following stage game that is repeated twice.

P_1 \ P_2	C	D	R
C	$\underline{2}, \underline{2}$	$\underline{6}, 1$	$0, 0$
D	$1, \underline{6}$	$5, 5$	$0, 0$
R	$0, 0$	$0, 0$	$\underline{4}, \underline{4}$

To sustain a cooperation in the second stage, players can play the strategy:

$$s_{i,2} = \begin{cases} R_{i,2} & \text{if } H^1 = (D, D) \\ C_{i,2} & \text{otherwise} \end{cases}$$

If a player deviate from cooperation (D, D) in stage 1, their total payoff would be $6 + 2 = 8$. If a player cooperate (D, D) in stage 1, their total payoff would be $5 + 4 = 9 > 8$. As such, this reward-punishment non-NE SPNE can be supported in this twice repeated game.

In general, in the case of multiple equilibria in the stage game, one can generally find some reward-punishment strategies that is **supportable** as an SPNE.

Now that we have gotten a taste of sustaining non-NE SPNEs in finitely repeated games, I hope you are excited to learn about all the possibilities in the infinitely repeated game world!

Remark: If a game is finitely repeated, you can think of it just as a really long extensive game. As such, SPNEs can generally be derived through backward induction. In the case of the motivating example prisoner's dilemma, we proved, by contradiction, that no cooperation is sustainable through backward induction.

Remark: Readers should also note that in the reward-punishment SPNE, the punishment must be an NE, because otherwise the player that is being punished can "deviate" to play the best response and the punishment would be in vain.

4.2 Infinitely Repeated Games

Consider, once again, our classic prisoner's dilemma game:

P_1 \ P_2	C	D
C	$\underline{2}, \underline{2}$	$\underline{6}, 1$
D	$1, \underline{6}$	$5, 5$

If this game is repeated infinitely many times, can we possibly get a non-NE SPNE?

The answer is YES! With the help of our old friend **discount rate** $\delta \in [0, 1)$.

Definition: Let the payoff of player i in stage t be denoted as $u_{i,t}$. The player's discounted payoff is: $u_i = \sum\limits_{t=0}^{\infty} \delta^t u_{i,t}$.

Recall that the NE (C, C) here is undesirable and (D, D) is Pareto optimal. If we follow the reward-punishment strategy introduced in finitely repeated games, we can write such strategy as:

$$s_{i,t} = \begin{cases} D_{i,t} & \text{if } H^{t-1} = (D, D) \\ C_{i,t} & \text{Otherwise} \end{cases}$$

Then, this strategy is sustainable as long as:

$$\underbrace{6 + \delta \cdot 2 + \delta^2 \cdot 2 + \cdots}_{u_j((C_{j,1},\ldots),(D_{i,1}),\ldots))} \leq \underbrace{5 + \delta \cdot 5 + \delta^2 \cdot 5 + \cdots}_{u_j((D_{j,1},\ldots,),(D_{i,t},\ldots))}$$

If you remember power series, you know that this means $\delta \in [0, 1)$ must satisfy:

$$6 + \frac{\delta}{1-\delta} \cdot 2 \leq 5 + \frac{\delta}{1-\delta} \cdot 5 \;\Rightarrow\; \delta \geq \frac{1}{4}$$

So any $\delta \in \left[\frac{1}{4}, 1\right)$ can sustain the infinite version of the reward-punishment strategy as an SPNE in this infinitely repeated game. This strategy is called the **Nash Reversion/Grimm Trigger Strategy**.

Definition: A cooperation strategy is called a **Nash Reversion/Grimm Trigger Strategy** if it rewards players that cooperate by furthering cooperation and punishes players who deviate by playing the "worst" Nash equilibrium for the rest of the game.

Definition: An infinitely repeated game is a stage game $\Gamma_N = \{\mathcal{I}, \{S_i\}, \{u_i\}\}$ that is repeated infinitely many times.

Specific Notations:

- $H^t(s_1, \ldots, s_I)$ is the history of play induced by the strategies $(s_1, \ldots, s_I)$
- $s^t(s_1, \ldots, s_I)$ is the outcome in period t induced by the strategies $(s_1, \ldots, s_I)$
- $s^\tau(s_1, \ldots, s_I)$ is the outcome in periods $t > \tau$ after history H^τ
- Aggregate payoff for player i under a given strategy profile is:

$$u_i(s_1, \ldots, s_I) = \sum_{t=1}^{\infty} \delta^{t-1} u_i(s^t(s_1, \ldots, s_I))$$

- The average payoff for player i of a strategy profile is $(1 - \delta)u_i(s_1, \ldots, s_I)$. This represents the "per period payoff" that has the same discounted total value as the payoff stream.
- The continuation payoff for player i following history H^τ under a given strategy profile is:

$$u_i(s_1, \ldots, s_I \mid H^\tau) = \sum_{k=0}^{\infty} \delta^k u_i(s^t(s_1, \ldots, s_I) \mid H^\tau)$$

  So when considering a deviation, the deviating payoff is not discounted, but then the future payoffs are.

Proposition: Consider any outcome a where $u_i(a) > u_i^{NE}$, $\forall i \in \mathcal{I}$. Then there exists a $\underline{\delta} \in [0, 1)$ such that $\forall \delta > \underline{\delta}$, the infinite repetition of outcome a is the outcome path of a SPNE with Nash Reversion in the repeated game.
Proof is currently omitted.

4.2.1 Infinitely Repeated Bertrand Competition

Consider the Bertrand price competition. The general SPNE result is that if there are more than 2 firms, all firms price at average cost and make 0 profit. As such, if the competition is repeated infinitely many times, it is natural to think that a collusion of some sort may be sustained.

Suppose there are 2 firms in the market. Can they collude to both price at a higher price so that they can make non-zero profit? Yes! One Nash reversion SPNE is to price at Monopoly price and split the monopoly profit. The strategy for firm i is as follows:

$$p_{i,t} = \begin{cases} p^m > c & \text{if} H^{t-1} = (p^m, p^m) \\ p = c & \text{Otherwise} \end{cases}$$

Then, to sustain this SPNE, we need to find $\delta \in [0, 1)$ such that

$$\underbrace{\pi^m}_{\substack{\text{Profit from} \\ \text{Deviating to} \\ p^m - \varepsilon}} + \underbrace{\frac{\delta}{1-\delta} \cdot 0}_{\substack{\text{NRS Punishment} \\ \text{is Competitive Profit}}} \leq \underbrace{\frac{\pi^m}{2} + \frac{\delta}{1-\delta}\frac{\pi^m}{2}}_{\substack{\text{Aggregate Profit} \\ \text{Under Collusion}}} \Rightarrow \delta \geq \frac{1}{2}$$

It can further be shown that any price between $(c, \bar{p})$ where $p = c$ is the competitive price and $\bar{p}$ is such that $x(\bar{p}) = 0$ can be sustained by some $\delta \in [0, 1)$. These were gone over in lecture but is omitted here.
Another possibility of SPNEs is a "switching strategy" where in each period, the players alternate on who charges the monopoly price and get monopoly profit from the market.

4.2.2 Infinitely Repeated Linear Cournot Competition

Consider our game of the Cournot quantity competition. Recall that the general competition results for a market with linear demand $q = a - q$ are that for J firms with marginal cost c, we have

- $J = 1$, $q^m = \frac{a-c}{2}$, $\pi^m = \frac{(a-c)^2}{4}$
- $J = 2$, $q_1 = q_2 = q^C = \frac{a-c}{3}$, $\pi^c(2) = \frac{(a-c)^2}{9}$

- $J \geq 2$, $q^C = \frac{a-c}{J+1}$

Now, recall that, in the Bertrand competition, the non-monopoly equilibrium price is always just the average cost, so intuitively, there is a "greater possibility" of collusion. So the natural question should be, can collusion happen in the Cournot game?

Suppose there are two firms in the market. In a non-repeated game, the unique SPNE is to produce $q = \frac{a-c}{3}$ so that each firm gets the profit $\frac{(a-c)^2}{9}$. However, if they choose to collude and both produce at $\frac{1}{2}q^m$, they can improve their profit to $\frac{(a-c)^2}{8}$.
Suppose this is the Nash Reversion so that each firm i plays:

$$q_{i,t} = \begin{cases} \frac{1}{2}q^m & \text{if } H^{t-1} = \left(\frac{1}{2}q^m, \frac{1}{2}q^m\right) \\ q = q^C & \text{Otherwise} \end{cases}$$

The discount rate δ needed in order to sustain this as an SPNE must satisfy:

$$\frac{(a-c)^2}{4} + \delta \cdot \frac{(a-c)^2}{9} + \delta^2 \cdot \frac{(a-c)^2}{9} + \cdots \leq \frac{(a-c)^2}{8} + \delta \cdot \frac{(a-c)^2}{8} + \delta^2 \cdot \frac{(a-c)^2}{8} + \cdots$$
$$\frac{(a-c)^2}{4} + \frac{\delta}{1-\delta}\frac{(a-c)^2}{9} \leq \frac{(a-c)^2}{8} + \frac{\delta}{1-\delta}\frac{(a-c)^2}{8}$$
$$\delta \geq \frac{9}{17}$$

So for any $\delta > \frac{9}{17}$, such NRS strategy is sustainable as an SPNE.

4.2.3 Folk Theorem

If you have been paying attention throughout this section, you probably have some doubts about *Nash Reversion Strategies.* Specifically, you likely wondered why we have only discussed such "harsh" strategies. If you were, you are in luck!

Consider the possibility of a "tit-for-tat" strategy" where you will cooperate unless one other player deviates, then you will deviate for one period only. Using our classical prisoners' dilemma game:

P_1 \ P_2	C	D
C	$\underline{2}, \underline{2}$	$\underline{6}, 1$
D	$1, \underline{6}$	$5, 5$

A tit-for-tat strategy in this game can be formally specified as:

$$s_i^t(H^{t-1}) = \begin{cases} C & , \text{ if } s_{-i}^{t-1} = C \\ D & , \text{ otherwise} \end{cases}$$

This strategy seems intuitive and can allow for future cooperation. However, it is not that simple because what if players just enter an infinite loop of vengeance? To answer that, we need to introduce some "regularity conditions".

Definition (Principle of Optimality): $(s_1, \dots, s_I)$ is an SPNE in the infinite repetition of a stage game if no player has a profitable one-stage deviation in any period after any history. Formally, this means $\forall i, t, H^t$,

$$u_i(s_i', s_{-i}^t(H^{t-1})) + \delta v_i(s_1, \dots, s_I \mid (s_i', s_{-i}^t(H^{t-1}))) \leq v_i(s_1, \dots, s_I \mid H^{t-1})$$

Instead of worrying about if any player will deviate for some $c \in \mathbb{N}$ times in the game, we can simply restrict it so that even deviating 1 time is not profitable, let alone c times.

Intuitively, this can save us a lot of work, but we do have to give up on some flexibility in which outcomes can be sustained. Luckily, we have a formal theorem telling us just what we are giving up, and how we can best utilize this result.

Definition (Feasible Payoffs): $(u_1, \dots, u_I)$ is **feasible** if it is a convex combination of pure-strategy payoffs of the stage game.

Theorem 4.1: Folk Theorem

As $\delta \to 1$, any feasible discounted average payoff $(1-\delta) \cdot (v_1, \dots, v_I)$ such that $\forall i \in \mathcal{I}$,

$$v_i > \min_{s_{-i}} \left\{ \max_{s_i} \{u_i(s_i, s_{-i})\} \right\}$$

can be sustained as a discounted average payoff in SPNE.

The right hand side here is commonly referred to the "minmax value". It is the lowest value that other players can force the player to get (as in forcing the other player to play their worst "best response").

In general, any convex combination payoffs that are "above" the minmax values can be supported. However, if a specific payoff is a minmax, you need to make sure that the player enforcing the punishment does not have a profitable deviation from the punishment (this is to make sure that the threat of punishment is credible).

For example, if Mahd deviated from a cooperation with Willy (which Mahd would never do because that is lying, and I think lying is Haram). The only supportable "punishment" Willy can place on Mahd is something that Mahd does not have profitable deviation from but Willy must also NOT have an incentive to deviate from the punishment.

Note: The payoff v described in Folk theorem is the "average payoff" calculated as (let v^t denote payoff in period t):

$$\begin{aligned}
v_i + \delta \cdot v_i + \delta^2 \cdot v_i + \cdots &= v_i^1 + \delta \cdot v_i^2 + \delta^2 \cdot v_i^3 + \cdots \\
\iff \frac{v_i}{1-\delta} &= v_i^1 + \delta \cdot v_i^2 + \delta^2 \cdot v_i^3 + \cdots \\
\iff v_i &= (1-\delta)(v_i^1 + \delta \cdot v_i^2 + \delta^2 \cdot v_i^3 + \cdots)
\end{aligned}$$

So whatever you solve as the total expected payoff from the SPNE, the average payoff (used in Folk theorem) is the total payoff multiplied by $(1-\delta)$.

5 Information Asymmetry

At the beginning of the semester, we talked about how Game theory is used to study what happens in the market when "perfectly competitive market" does not hold anymore. Among the conditions required for perfect competition is perfect information. Finally, now is the time we get to study what happens when there is information friction in the market!

In general, we will split up the discussion into **Decentralized Outcomes** and **Centralized Outcomes**. Decentralized Outcomes is the conventional approach where we create thought experiments to study the market outcome in the presence of asymmetric information. In general, and this should come as no surprise, the market will have inefficiencies, and the loss can be so strong that the market may collapse. One thing to note is that such inefficiencies cannot be corrected by exogenous government actions like transfers. This unfortunate result is called **Adverse Selection**, and the market solution for it is called **Signalling** (solution in terms of keeping the market from collapsing). In signaling, the player(s) with more information can costly signal their private information, leading to **separating equilibrium** where players' private information is basically public and **pooling equilibrium** where information asymmetry remains.

Centralized Outcomes takes a novel approach to the information asymmetry problem. The idea is that, instead of the more informed party bearing the cost to try to correct for the inefficiencies, what if the less informed party takes action? We will discuss **Moral Hazard** (The design of contract by the less informed party to elicit actions from the more informed party), **Monopolistic Screening** (The design of contact to elicit information from the more informed party), and **Mechanism Design** (The design of a game by the less informed party to elicit both information and actions from the more informed parties).

I hope you are as excited as I am to learn about this topic. However, one must learn to differentiate before one knows how to integrate, so let's begin with some fundamentals!

5.1 Bayes Nash Equilibrium

As a motivating example, consider the following matching pennies game between Alex and Mahd. In this game, Alex always prefers to match with Mahd, but Mahd may or may not prefer to match with Alex. Suppose that Mahd is type $M1$ (wants to match) with probability p and type $M2$ (wants to not match) with probability $1-p$. The payoff tables are as follows:

Alex \ M_1	a_1	b_1
a	$\underline{3}, \underline{1}$	$0, 0$
b	$0, 0$	$\underline{1}, \underline{3}$

Alex \ M_2	a_2	b_2
a	$\underline{3}, 0$	$0, \underline{1}$
b	$0, \underline{3}$	$\underline{1}, 0$

If Mahd is M_1, then there are clearly 2 Pure Strategy Nash Equilibria, but if Mahd is M_2, then there can only be a mixed-strategy Nash Equilibrium. Now suppose Alex does not know Mahd's type, but Mahd does. What is the Nash Equilibrium in this game?

Definition: A **Bayesian Game** is a game $\Gamma_N = \{\mathcal{I}, \{S_i\}, \{u_i\}, \Theta, F\}$ where Θ is the collection of player types and F is the distribution of types.

Definition: A **Reformulated Bayesian Game** is a game $\Gamma_N = \{\mathcal{I}, \{\prod_{\theta_i} S_i^\theta\}, \{\tilde{u}_i\}\}$ where each player of n types is treated as a player with n information sets.

The reformulated Bayesian game simplifies the game for the less informed party. Essentially, instead of thinking about what nature may "play", the less informed party can evaluate their strategies based on the average payoffs calculated using the probability of their rival's types and the induced strategies. As such, a Bayesian game is simplified to a Normal game with slightly more complicated actions, and hence we can find the NEs in the Bayesian game just like how we find NEs in games of perfect information. These NEs are called **Bayes Nash Equilibrium**.

The caveat here is that the less informed players calculate their expected payoffs using subjective probabilities that may or may not be correct. If the probabilities are objective and known to all players a priori, then everything is fine and dandy, because then the difference between subjective and objective probabilities play no role. However, if the only the subjective probabilities are available to the less informed player, then we need to go back to *Perfect Bayesian Equilibria* to make sure that the beliefs (a.k.a. subjective probabilities) are consistent (with the objective probabilities).

First let's look at the benchmark cases when the probabilities are objective.

5.1.1 Simple Auctions

Consider a **first-price sealed-bid** auction where n bidders bid for an object. In this auction, the highest bidder wins and pays their bid. Each bidder's valuation v_i of the object is drawn independently from the CDF $F(v)$ with $supp(v) = [\underline{v}, \bar{v}]$. If there is a tie, one of the bidder is selected randomly and fairly to win the auction (but we will soon see why that practically does not matter).

As described, the payoff of bidder i is:

$$u_i(b_i, b_{-i}) = \begin{cases} v_i - b_i & , \text{ if } b_i > \max\{b_{-i}\} \\ \frac{1}{2}(v_i - b_i) & , \text{ if } b_i = \max\{b_{-i}\} \\ 0 & , \text{ otherwise} \end{cases}$$

Intuitively, we know that the bidder's bid should be a strictly increasing function of her valuation (If it is not, bidding would make a lot less sense). As such, we can write i's bid as $b_i = b(v_i)$. For simplicity, let's guess $b(v_i) = cv_i + \underline{v}$ is the symmetric BNE. Bidder i's expected value in this auction is thus:

$$P(b_i > b_j)(v_i - b_i) + P(b_i = b_j)\frac{1}{2}(v_i - b_i)$$

We then want to check for deviation[16]. Since $b(\cdot)$ is a strictly increasing function in v_i, there exists an inverse function $b^{-1}(\cdot)$ mapping bids to valuations, so the expected payoff is:

$$P(b^{-1}(b_i) > \max\{b^{-1}(v_{-i})\})(v_i - b_i) + P(b^{-1}(b_i) = \max\{b^{-1}(v_{-i})\})\frac{1}{2}(v_i - b_i)$$

Now, $b^{-1}(b_j) = v_j$ because it maps whatever bid j makes to j's valuation, so the payoff is really:

$$P(b^{-1}(b_i) > \max\{v_{-i}\})(v_i - b_i) + P(b^{-1}(b_i) = \max\{v_{-i}\})\frac{1}{2}(v_i - b_i)$$

Since v_j is a continuous random variable, $P(b^{-1}(b_i) = v_j) = 0$. Since i wants to maximize their payoff, they have to solve:

$$\max_{b_i} F(b^{-1}(b_i))^{n-1}(v_i - b_i)$$

[16] In BNE, players should not want to deviate. So we can solve for c through the expected value maximization problem.

The F.O.C. is[17]

$$(n-1)F^{n-2}(b^{-1}(b_i))\frac{1}{b'(v_i)}f'(v)(v_i-b_i)-F^{n-1}(b^{-1}(b_i))=0$$

Our goal is to solve for $b(v_i)$, so we can rewrite the F.O.C. as:

$$(n-1)F^{n-2}(b^{-1}(b_i))f'(v)\boldsymbol{b_i}+F^{n-1}(b^{-1}(b_i))\boldsymbol{b'(v_i)}=(n-1)F^{n-2}(b^{-1}(b_i))f'(v)\boldsymbol{v_i}$$

In equilibrium, we must have $b_i=b(v_i)$, so $b^{-1}(b_i)=v_i$. Notice that LHS is just the derivative (using product rule) of $F^{n-1}(v_i)b(v_i)$ with respect to v_i, so we can rewrite the F.O.C. as:

$$\begin{aligned} dv_i\cdot\frac{d}{dv_i}F^{n-1}(v_i)b_i(v_i) &= (n-1)F^{n-2}(v_i)\underbrace{f'(v_i)}_{=\frac{dF}{dv}}v_i\cdot dv_i \\ \Rightarrow\ dF^{n-1}(v_i)b_i(v_i) &= (n-1)F^{n-2}(v_i)v_i\,dF \end{aligned}$$

Take the indefinite integral (using integration by parts) we get

$$\begin{aligned} \int 1\,dF^{n-1}(v_i)b_i(v_i) &= \int \underbrace{(n-1)F^{n-2}(v_i)}_{dv}\underbrace{v_i}_{u}\,dF \\ \iff F^{n-1}(v_i)b(v_i) &= F^{n-1}(v_i)v_i-\int_{\underline{v}}^{v_i}F^{n-1}(v)dv \end{aligned}$$

So the symmetric BNE bidding strategy in this game is:

$$b(v_i)=v_i-\int_{\underline{v}}^{v_i}\frac{F^{n-1}(v)}{F^{n-1}(v_i)}dv$$

If $v_i\sim Uniform[0,1]$ (so $\underline{v}=0$, $\bar{v}=1$), then we can write

$$b(v_i)=v_i-\int_{\underline{v}}^{v_i}\frac{F^{n-1}(v)}{F^{n-1}(v_i)}dv=v_i-\int_{\underline{v}}^{v_i}\frac{v^{n-1}}{v_i^{n-1}}dv=v_i-\frac{1}{n}\frac{v_i^n}{v_i^{n-1}}=b_i-\frac{v_i}{n-1}=\frac{n-1}{n}v_i$$

When $N=2$ and $v_i\overset{iid}{\sim}U[0,1]$, we have $b(v_i)=\dfrac{v_i}{2}$.

[17] To see how to transform the derivative of b^{-1}, look up inverse function derivative on the internet. Wikipedia has a pretty good walk-through of it but I would recommend that you try to follow the derivation and do it yourself so you don't forget.

Example: All-Pay Auction

Consider a sealed-bid, **all-pay** auction for a single item, in which bids are selected from the continuum. There are two risk-neutral bidders, whose utilities are quasi-linear in money. Bidder i's valuation, denoted v_i, for the item, is private information and the $v_i(i = 1, 2)$ are i.i.d. random variables that are each uniformly distributed on the interval [1,2].

After observing her own valuation v_i, each bidder simultaneously and independently submits a sealed-bid $b_i \in [0, 2]$ for the item. The highest bidder wins the item. However, since this is an all-pay auction, every bidder must pay the amount of her bid, regardless of whether she wins or loses. If $b_j > b_k$ then bidder j wins the item; but bidder j pays b_j and bidder k pays b_k. Ties are resolved by fair randomization.

Solve for a symmetric Bayesian-Nash equilibrium of this game.

Solution: Suppose that both players' bids are identical strictly increasing function $b(v_i)$ in valuation. Player i solves the problem

$$\max_{b_i} P(b_i > b_j)v_i - b_i \equiv P(b^{-1}(b_i) > v_j)v_i - b_i$$

The first order condition with respect to b_i is

$$\underbrace{\frac{\partial P(b^{-1}(b_i) > v_j)}{\partial b^{-1}(b_i)}}_{\text{Derivative of CDF=PDF}} \underbrace{\frac{\partial b^{-1}(b_i)}{\partial b_i}}_{=\frac{1}{db/dv}} v_i - 1 = 0 \;\Rightarrow\; 1 \cdot \frac{1}{b'} v_i = 1 \;\Rightarrow\; b = \frac{1}{2}v_i^2 + C$$

Now, given the valuation of 1, the expected payoff is 0, so the bid should also be 0. As such, we must have

$$\frac{1}{2}1^2 + C = 0 \;\Rightarrow\; C = -\frac{1}{2}$$

So the BNE bidding function is:

$$b(v_i) = \frac{1}{2}v_i^2 - \frac{1}{2}$$

Example: Prelim 2022 SS Q3

A seller wants to sell a single object and has two interested buyers. The buyers' valuations of the object are independently and uniformly distributed on $[0, 1]$, and each buyer privately knows her own valuation. The seller's value of the object is 0.
Suppose the seller adopts the following selling strategy:

She approaches one of the buyers (chosen at random) and makes a take-it-or-leave-it offer at a fixed price p_1. If the first buyer accepts the offer, the object is sold to him at the offered price. If the first buyer declines the offer, the seller then approaches the other buyer with a take-it-or-leave-it offer at a fixed price p_2. If the second buyer accepts the offer, the object is sold to him at the offered price. If neither buyer accepts, then the seller keeps the object.

(a) Derive the optimal values of p_1 and p_2. What is the seller's expected revenue from this selling strategy?

Using backward induction, when facing the second buyer, the seller solves the problem

$$\max_{p_2} \; p_2 \cdot \underbrace{[1 - F(p_2]}_{P(v_2 \geq p_2)} \equiv p_2(1 - p_2) \;\Rightarrow\; p_2^* = \frac{1}{2}$$

This means that the seller's expected payoff is $\frac{1}{4}$ if they get to the second buyer. So when the seller goes to the first buyer, they solve the problem

$$\max_{p_1} \; p_1 \cdot \underbrace{[1 - F(p_1]}_{P(v_1 \geq p_1)} + \underbrace{\frac{1}{4}}_{\substack{\text{Expected}\\\text{Profit}\\\text{from}\\\text{buyer 2}}} \cdot \underbrace{[F(p_1]}_{P(v_1 < p_1)} 1 - p_2) \;\Rightarrow\; p_1^* = \frac{5}{8}$$

So in this strategy, the seller would price $p_1^* = \frac{5}{8}$, $p_2^* = \frac{1}{2}$ with the total expected profit $\frac{25}{64}$. This result is fairly intuitive. The seller can take more risk and sell at a higher price to buyer 1 because the seller has a backup buyer then. But when facing buyer 2, the seller has to sell at a more "fair" price.

(b) If the seller ran a first-price auction (without reservation price), what would be the equilibrium bidding strategies and the seller's expected revenue?

As discussed before, in the $n = 2$ case, the BNE bidding strategy is $b(v_i) = \frac{v_i}{2}$. So the distribution of the winning bid v is

$$P(\max v_i, v_j \leq v) = P(v_i \leq v)P(v_j \leq v) = v^2 \Rightarrow PDF : 2v$$

So the seller's expected payoff is:

$$E\left[\frac{v}{2} \mid \max\{v_i, v_j\} \leq v\right] = \int_0^1 \frac{v}{2} \cdot 2v \, dv = \frac{1}{3}$$

(c) Between the two selling strategies, which one is more efficient (in terms of allocative efficiency) and which one does the seller prefer? Does the seller always prefer the more efficient option?

In the first strategy, it can happen buyer 2 bought the good because her reservation price is $\frac{1}{2}$ while buyer 1's reservation price is $\frac{1}{2} + \varepsilon$, but buyer 2 is the one winning the good, so it is not allocatively efficient. Intuitively, the seller is basically using price discrimination to maximize profit, so there is dead-weight-loss in the market.

On the other hand, the seller wants to maximize profit by choosing the first strategy because the expected profit is higher. It makes sense that the seller is not necessarily thinking about allocative efficiency, but rather they just want to profit maximize. As such, in the imperfect market, sellers can extract more surpluses and create dead-weight-loss at the expense of the consumers.

5.2 Adverse Selection

The pioneer of the adverse selection study is the "Lemons" model (Akerlof 1970 QJE). Consider a used car market where there are N sellers and an arbitrarily large amount of buyers. Each seller knows the quality of the car θ, but buyers only know that $\theta \sim U[0, 1]$. For a car of quality θ, the seller has reservation value $\frac{2}{3}\theta$, and the buyer is willing to pay θ.

Under perfect information, buyers would know the exact quality, and pay $p = \theta$. However, if the buyers only know that $\theta \sim U[0, 1]$, they would not want to pay more than the expected quality $p \leq 1$ for a car. So only sellers with cars $\frac{2}{3}\theta \leq p \Rightarrow \theta \leq \frac{3}{2}$ would sell. Knowing this, the buyer would only buy if

$$E\left[\theta \,\middle|\, \theta < \frac{3}{2}p\right] = p \;\Rightarrow\; \frac{3}{4}p = p \;\Rightarrow\; p = 0$$

As such, buyers would not buy any cars with strictly positive prices, and the market would collapse (a.k.a. complete unraveling of the market). This model may seem simple and straightforward, but it actually gives us a lot of insights into how asymmetric information inefficiencies can potentially be fixed. One application of this model is in the labor market.

> (Prelim SS 2023)
>
> Suppose that the seller's reservation price is $k\theta$ where $k \in (0, 1)$, what is the maximum k such that the market does not unravel?
>
> If there is a car testing supplied for a non-zero price, can you intuitively tell whether the seller or the buy, or both, will pay for testing in equilibrium?

5.2.1 Labor Market with Asymmetric Information

Consider a labor market with only one firm acting as a monopsony with the following set-up:

- Workers know their marginal productivity $\theta \in \{\theta_L, \theta_H\}$
- Firm does not know θ, but knows $\theta \sim F(\theta)$
- A worker with marginal productivity θ has reservation wage $r(\theta)$ where $r(\theta)$ is strictly increasing and $r(\theta) \leq \theta$
- Assume that $E[\theta \mid \theta \in \varnothing] = \theta_L$

Definition (CE): A wage and type-set tuple (w^*, Θ^*) is a **competitive equilibrium** if

(i) $\Theta^* = \{\theta \mid r(\theta) \leq w^*\}$

(ii) $w^* = E[\theta \mid \theta \in \Theta^*]$

In words, it means that in a competitive equilibrium, workers who work are paid the average marginal productivity of those who would work (think of it like how you would the average cost of a production input).

Notice that in condition (ii), the average marginal productivity of those who would work (RHS) is a function increasing and continuous in w^* that becomes a constant at $w = E[\theta]$ (because firms would not want to pay more than $E[\theta]$ for a worker who is expected to have marginal productivity $E[\theta]$). The equilibrium[18], then, must be where this function intersects the 45-degree line $w = \theta$. Since we don't know the exact functional form of $E[\theta \mid \theta \in \Theta^*]$, there can be just a unique CE or multiple CEs. In the case of multiple CEs, they are Pareto-ranked with the higher the wage/productivity the better.

Suppose now there are $\underline{N = 2 \text{ firms}}$ in the market so that they are in a Bertrand price competition for workers. Assuming that:

- All the assumptions from above
- Workers either accept the higher offer or reject both offers
- Let W^* denote the set of all competitive equilibrium wages and $w^* = \max\{w \mid w \in W^*\}$ the Pareto-dominating competitive wage.
- Assume $w^* > r(\theta_L)$
- Assume that $\exists \varepsilon > 0$ such that $E[\theta \mid r(\theta) \le w'] > w'$, $\forall w' \in (w^* - \varepsilon, w^*)$, that is, w^* is the maximum wage and the expected payoff cuts the 45-degree line from above. It eliminates the case for $w = E[\theta]$.

Surprisingly, not a whole lot would change, because the one firm was already hiring with "zero profit" as they set the wage to the expected marginal productivity. In the Bertrand competition, we know that the prices would be set so that there is no profit. The actual equilibrium levels, in this case, are the same as the one-firm case, and $w_1 = w_2 = w^*$.

In this setting, the firms are pretty much stuck with doing the guesswork and can only offer wages conservatively. If you have been doing these thought experiments in your head, you probably thought "Why wouldn't the higher productivity workers just come out and say they are more productive?" You are absolutely right, they totally should. However, giving out information about your private type must come at a cost (maybe all your friends would hate you for being smart or something. Kidding. If revealing one's true type does not come at a cost, low productivity workers would simply lie, and we would be back to the asymmetric information problem). This leads us to our next topic - **Signalling**.

[18] An important thing to make clear is that competitive equilibria exist in more than just competitive markets. It is a common misunderstanding among students that competitive equilibria implies perfectly competitive markets (perhaps due to the phrasing). If you find yourself confused by this statement, you should review 812A notes on Walrasian Equilibrium.

5.3 Signaling

The idea here is simple. Both the firms and the workers of higher productivity (call this high-type or H from here on) can be better off if the firms know they are more productive. But to make sure that only high-type "signals" (self-identifies), the firm must make it such that it is only worth it for high-types to self-identify (so it must make low-type worse off if they were to lie about being high-type). One intuitive application of this is using education as a costly signal. There are 2 main types of models here: **unproductive signals** and **productive signals**.

Spence (Unproductive Signals): Education may not have intrinsic value since it is only used for the purpose of signaling. In that spirit, getting a Master of Education may only help the teacher get paid more due to the signal, even if they did not become better teachers (This is a real-world example that hits deep, I am sure that Salem Rogers would love to chat with you about it, shall you be interested).

Becker (Productive Signals): Education increases a worker's productivity as well as their perceived productivity (type). There can hence be an externality in getting higher education as you can be paid more than other high-type workers with lower education.

In practice, both views can seem a little extreme and is almost always on a case-by-case basis. For the most part, we would expect to see a mix of both, where education is not useless but may not be as productive as we want it to be (and happily recall that Steven Levitt once said "If you are smart enough to get a Ph.D. in Economics, you are probably smart enough to know not to get a Ph.D. in Economics.")

5.3.1 Spence Model (1973 QJE)

The model has the following setup:

- There are two types of workers $\theta_L, \theta_H > \theta_L$ with $P(\theta = \theta_H) = \lambda \in (0, 1)$
- Outside options for both types are normalized so that $r(\theta_L) = r(\theta_H) = 0$
- $e \in [0, \infty)$ is the education level that is totally unproductive
- $c(e, \theta)$ is the cost for type θ worker to get education level e and
  - $c(0, \theta) = 0$
  - $c_e > 0$, $c_{ee} > 0$, $c_\theta < 0$ for $e > 0$

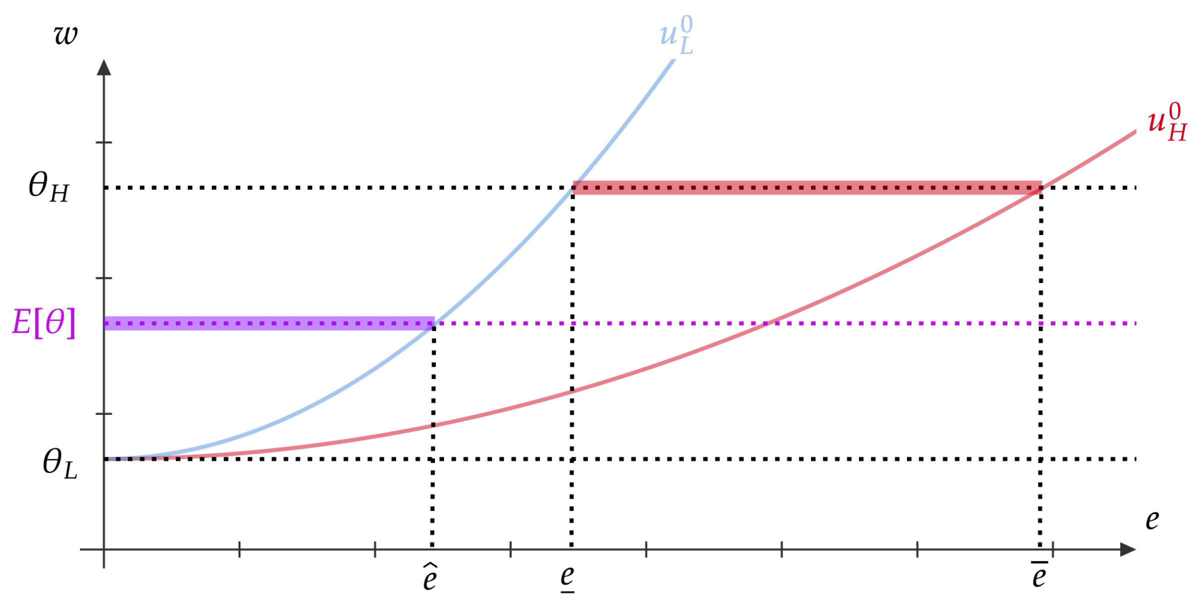
  - **Single Crossing Condition:** $c_{e\theta} < 0$

- Worker utility is $u = w - c(e, \theta)$
- Firm profit is $\theta - w$

Since firms only have information about how worker types are distributed, we need to think about how they should make wage offers. Ideally, they want to offer high-type workers θ_H and low-type workers θ_L, but in the absence of perfect information, they can only offer some wage $w \in [\theta_L, \theta_H]$. Naturally, firms would think about the **"base case"**, which is just paying everyone θ_L, and see how to improve from there.

Consider Figure 5.3.1 where u_H^0 and u_L^0 represent the indifference curves of each worker-type if they simply get wage θ_L by showing 0 effort/education. Intuitively, you should recognize that being on such a wage schedule is inefficient, as firms would have wanted the high-type workers to perform with higher performance (paying wages in bananas will only attract monkeys), and high-type workers would have preferred to be paid to their abilities.

Figure 5.3.1: PBE Candidates



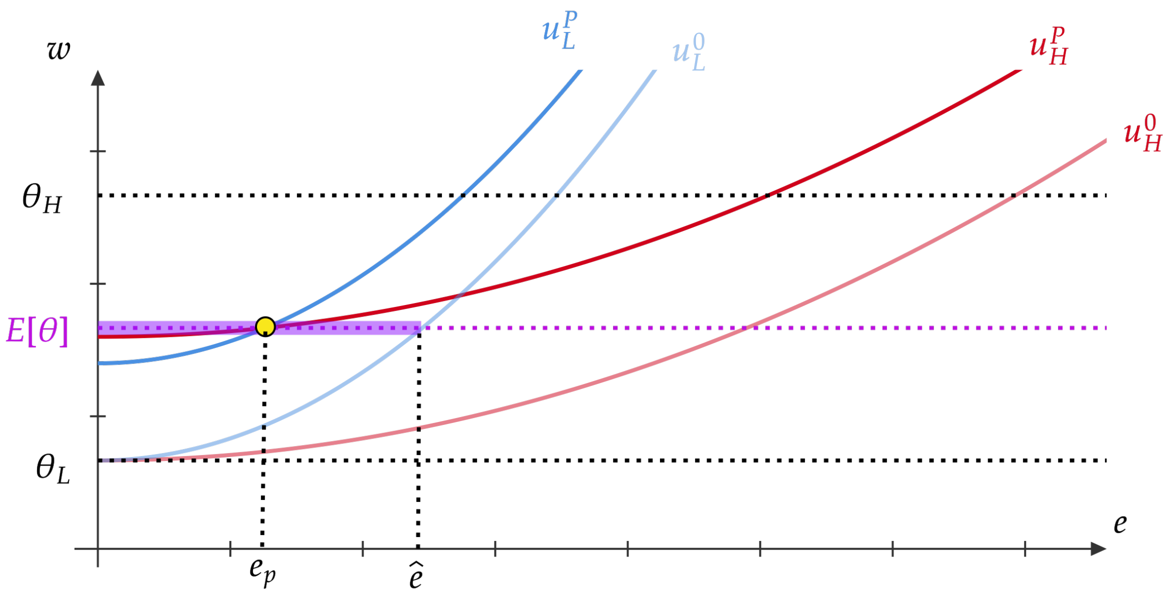
Notice that there are essentially two options here.

(i) Both workers exert some identical effort level $e_p \in [0, \hat{e}]$, and since firms cannot distinguish between the two, firms pay $E[\theta]$. This is called the **<u>Pooling</u> Perfect Bayesian Equilibrium**. In Figure 5.3.1, both types of workers will achieve higher utility from the entire purple highlighted wage schedule than the *base case*. We shall thus discuss those effort levels $e \in [0, \hat{e}]$.

(ii) High-type workers exert some high effort $e_H \in [\underline{e}, \bar{e}]$ such that low-type workers cannot exert e_H and be better off. Knowing this, firms will pay θ_H to workers with $e = e_H$ and θ_L to workers with $e = e_L$. This is called the **Separating Perfect Bayesian Equilibrium**. In Figure 5.3.1, both types of workers will achieve higher utility from the entire purple highlighted wage schedule than the *base case*. We shall thus discuss those effort levels $e \in [\underline{e}, \bar{e}]$.

5.3.2 Pooling PBE:

Figure 5.3.2: An Example of Pooling PBE



Consider the effort level e_p in Figure 5.3.2. This is a pooling PBE. Since both types are exerting the same effort e_p, firms are unable to tell workers apart, but they have a prior belief (based on their knowledge of the distribution of worker types) that λ of the workers are high-type. Firms would pay $E[\theta \mid e = e_p] = \lambda\theta_H + (1 - \lambda)\theta_L$ to workers who exert e_p.

At this wage schedule, neither workers (indifference curve u_H^P, u_L^P) have the incentive to exert more effort, and low-type workers have more incentive to exert a little bit less effort than high-type workers. So firms' beliefs would be consistent if they think anyone not exerting e_P is a low-type worker. The *Pooling PBE* can thus be formally described by:

Workers' Strategy:

$$e(\theta) = e_P, \forall\theta$$

Firms' Strategy (Wage Schedule):

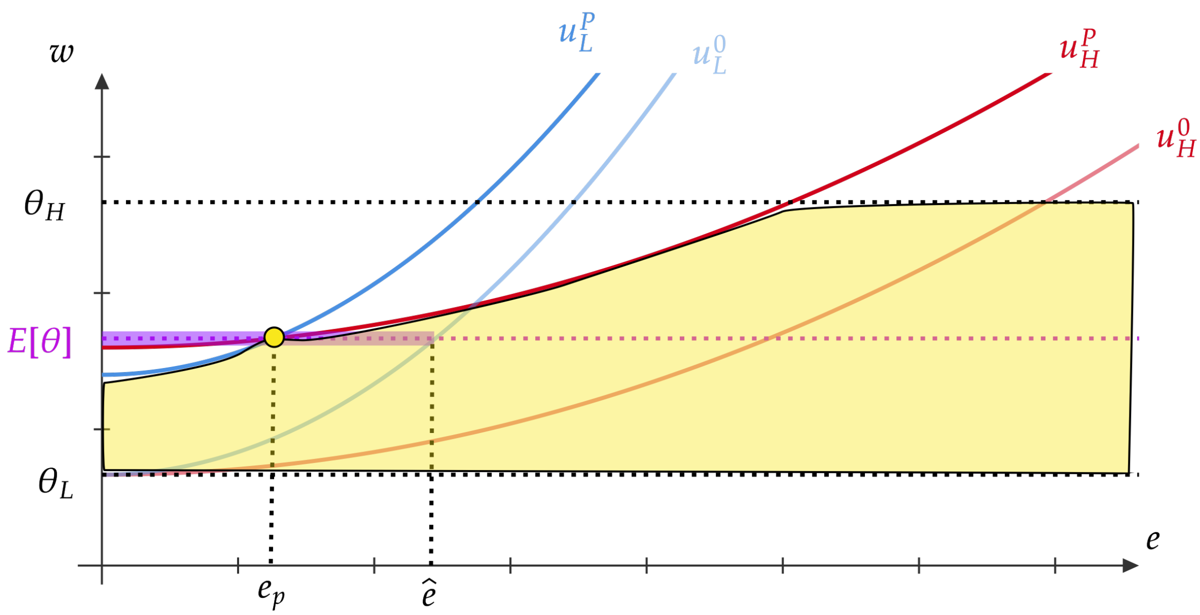
$$w(e) = \begin{cases} \lambda\theta_H + (1-\lambda)\theta_L & , \text{ if } e = e_p \\ w(e) & , \text{ if } e \neq e_p \end{cases}$$

Firms' Belief:

$$\mu(\theta = \theta_H \mid e = e_p) = \lambda$$
$$\mu(\theta = \theta_H \mid e \neq e_p) = 0$$

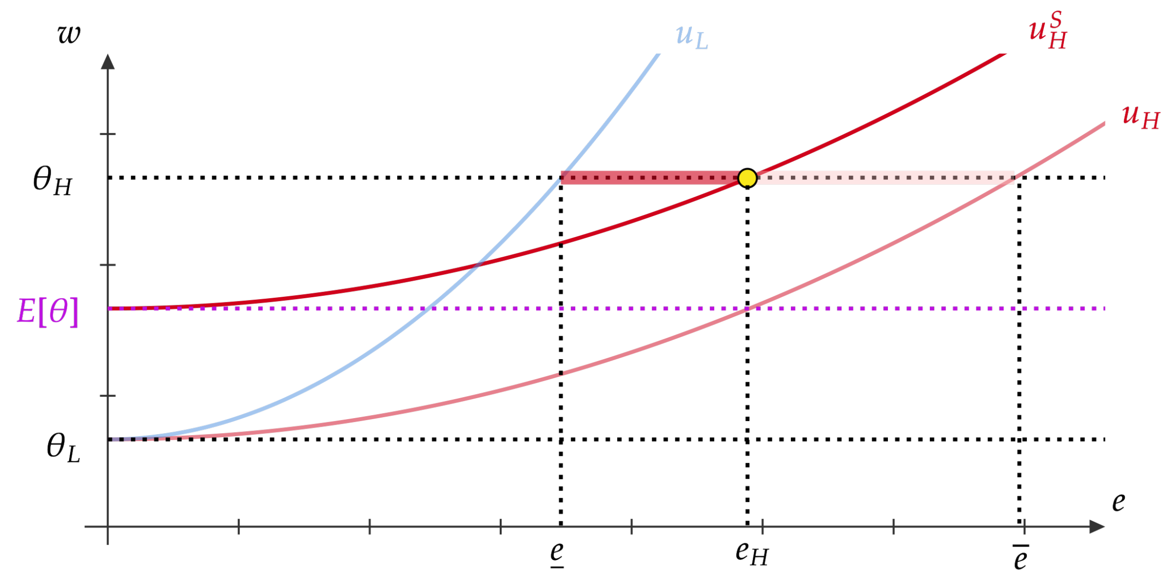
Notice that, to make sure that workers have no incentive to deviate, the off-path wage schedules must be bounded above by the $\min\{u_L^P, u_H^P, \theta_H\}$. In Figure 5.3.3, any wage schedule such that $w(e_p)$ is the yellow dot and $w(e \mid e \neq e_p)$ is in the yellow area is a sustainable strategy for the Pooling PBE described above.

Figure 5.3.3: Potential Wage Schedule of A Pooling PBE



5.3.3 Separating PBE:

Figure 5.3.4: An Example of Separating PBE



Consider the effort level e_H in Figure 5.3.4. This is a separating PBE. Since low-type workers will only be worse off by exerting e_H and getting θ_H, firms know that only high-type workers will exert e_H, and thus will pay those workers θ_H, making the high-type workers better off.

At this wage schedule, neither workers (indifference curve u_H^S, u_L) have the incentive to exert more effort. So firms' beliefs would be consistent if they think anyone not exerting e_H is a high-type worker and a worker is low-type otherwise. The *Separating PBE* can thus be formally described by:

Workers' Strategy:

$$e(\theta) = \begin{cases} e_H & \text{, if } \theta = \theta_H \\ e_L & \text{, if } \theta = \theta_L \end{cases}$$

Firms' Strategy (Wage Schedule):

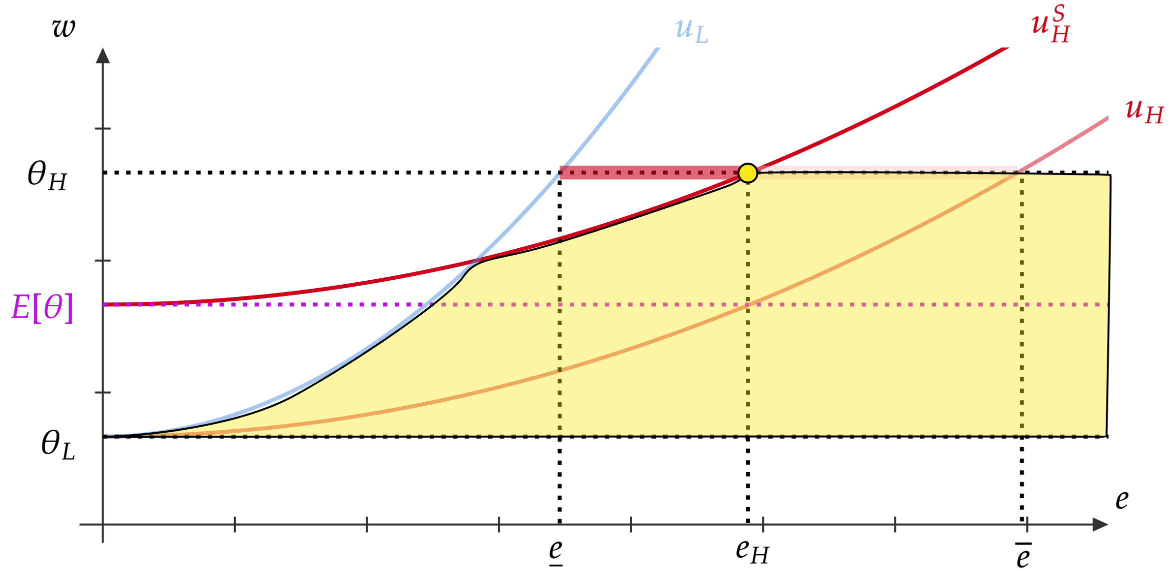
$$w(e) = \begin{cases} \theta_H & \text{, if } e = e_H \\ w(e) & \text{, otherwise} \end{cases}$$

Firms' Belief:

$$\mu(\theta = \theta_H \mid e = e_H) = 1$$
$$\mu(\theta = \theta_H \mid e \neq e_H) = 0$$

Similar to the pooling case, to make sure that workers have no incentive to deviate, the off-path wage schedules must be bounded above by the $\min\{u_L^P, u_H^P, \theta_H\}$. In Figure 5.3.5, any wage schedule such that $w(e_H)$ is the yellow dot and $w(e \mid e \neq e_p)$ is in the yellow area is a sustainable strategy for the Separating PBE described above.

Figure 5.3.5: Wage Schedule of A Separating PBE



5.3.4 Belief Refinement

Notice that in the pooling case from 5.3.3, since any effort level $e \in [0, \hat{e}]$ is technically sustainable as a pooling PBE, we want to see if there is any way to differentiate/judge them. Since workers will do $E[\theta]$ of work when on this purple band, it does not matter to the firm where the pooling effort level actually occurs (due to the Spence model assumption that efforts are totally unproductive[19]). As such, the most efficient pooling effort, in this case, will be $e = 0$, and workers of both types get paid $E[\theta]$.

Similarly, in the separating case from 5.3.5, since any effort level $e_H \in [\underline{e}, \bar{e}]$ is technically

[19] If the signals are productive (as in the Becker model), then the most efficient pooling one might be some non-zero level depending on the exact parameter values and functional forms.

sustainable as a separating PBE, we want to see if there is any way to differentiate/judge them, and the most efficient separating effort, in this case, will be $e_H = \underline{e}$. These refinements are logical, but we still need a way to formalize them:

Let Θ be the set of all possible types. For any given effort e and a subset of types $\widehat{\Theta} \subseteq \Theta$, define the set of wages that are consistent with some belief system $\mu(\cdot \mid e)$ that only takes strictly positive probabilities on the set $\widehat{\Theta}$ as

$$W^* \left(\widehat{\Theta}, e\right) \equiv \left\{ w \middle| \exists \mu(\theta \mid e)\, s.t.\, \mu(\theta \mid e) > 0 \text{ if } \theta \in \widehat{\Theta} \,\text{and}\, w = \int \theta d\mu(\theta \mid e) \right\}$$

Definition: Take some wage schedule set W^*. Effort e is **strictly dominated** for type θ if $\exists e' \neq e$ such that

$$\inf \left\{ W^*(\Theta, e') - c(e', \theta) \right\} > \sup \left\{ W^*(\Theta, e) - c(e, \theta) \right\}$$

So if someone of type θ can improve by deviating from e, then e is strictly dominated.

Definition (Reasonable Type Set): The type set $\Theta^*(e)$ is a reasonable type set if it assigns strictly positive probability to type θ if and only if e is not strictly dominated for type θ. For example, the reasonable type set of e_H in Figure 5.3.4 is:

$$\Theta^*(e_H) = \begin{cases} \{\theta_H\} & , \text{ if } e \in (\underline{e}, e_H] \\ \{\theta_H, \theta_L\} & , \text{ if } e \in [0, \underline{e}] \\ \varnothing & , \text{ otherwise} \end{cases}$$

Definition: A PBE has a **reasonable belief** if firms believe that a worker is not choosing a strictly dominated action. With reasonable beliefs, we can eliminate most of the pooling/separating PBEs other than $e_p = 0$ in Figure 5.3.3 and $e_H = \underline{e}$ in Figure 5.3.5. AND if the pooling indifference curve for high-type intersects some separating non-most-efficient PBE (exactly the case of Figure 5.3.5), then the most efficient pooling PBE is also strictly dominated. Note that this need not be the case and should always be checked).

5.3.5 Equilibrium Dominance and the Cho-Kreps Intuitive Criterion (1987)

One question you may have had in your mind so far, is "Why would high-type workers ever not want to distinguish themselves?" If you did have this question, good job! Because that is exactly why we need to introduce an even stronger refinement concept called **equilibrium dominance**.

Definition: An effort level e is **Equilibrium Dominated** for type θ if

$$u^*(\theta) > \sup \left\{ W^*(\Theta, e) - c(e, \theta) \right\}$$

where $u^*(\theta)$ is equilibrium payoff of type θ.
Definition: A type set $\Theta^{**}(e)$ is called an **Equilibrium Reasonable Set** if

$$\Theta^{**}(e) \equiv \{\theta \mid e \text{ is not equilibrium dominated for } \theta\} \tag{1}$$

Definition: The PBE (σ, μ) violates the **Cho-Kreps Intuitive Criterion** if there exists e' and θ such that

$$u^*(\theta) < \inf \left\{ W^*(\Theta^{**}(e'), e') - c(e', \theta) \right\}$$

Remark: Equilibrium dominance itself is not a further refinement of PBEs. Rather, we use equilibrium dominance to define the equilibrium reasonable set that defines Cho-Kreps. Notice that *strict dominance* compares utility with a type set that does not interact with the actions themselves. On the other hand, *Cho-Kreps* compare utilities with an updated belief about the type set that is restricted by specific actions. That being said, the difference is NOT the same as the difference between wPBE and PBE.
Remark: The *intuitive criterion* removes all **inefficient** separating PBEs and **all** pooling PBEs.

5.4 Monopolistic Screening

If you are a high-type worker, you might have been wondering "Why do I have to do more work just to show that I am not low-type?" If you have that thought, good for you, because that is what monopolistic screening is all about! Consider the following labor market:

- One firm and one worker
- The worker is either θ_L or $\theta_H > \theta_L$. $P(\theta = \theta_H) = \lambda$
- The worker has reservation utility $r(\theta_L) = r(\theta_H) = 0$
- The worker chooses effort $e \in [0, \infty)$ that is observable to all

- $c(e, \theta)$ is the cost for type θ worker to get education level e and
  - $c(0, \theta) = 0$
  - $c_e > 0$, $c_{ee} > 0$, $c_\theta < 0$ for $e > 0$
  - **Single Crossing Condition:** $c_{e\theta} < 0$
  - Worker utility is $u = w - c(e, \theta)$
  - Firm profit is $\pi(e) - w$. Assume $\pi'(e) > 0$, $\pi''(e) < 0$, and $\pi'(0) > c_e(0, \theta)$.

5.4.1 Perfect Information and First Best Contracts

Since the firm is a monopolist here, we know they are profit-maximizing. Under perfect information, the monopolist can pay different wages (w_L, w_H) for different effort levels (e_L, e_H), and hence extract all worker surpluses, making the utility of all types of workers $u_L = u_H = 0$. These contracts are what we call the **first-best** contracts[20].

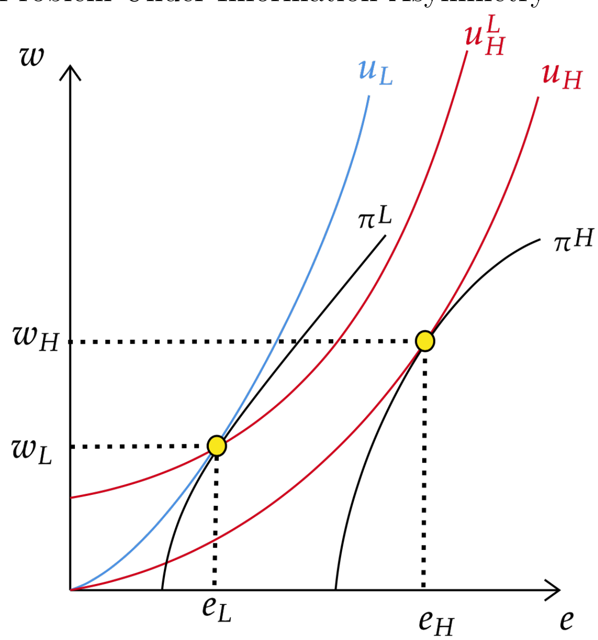
Figure 5.4.1 illustrates how the firm uses different contracts $((w_L, e_L), (w_H, e_H))$ to extract all worker surpluses by making sure both workers are indifferent between working at that wage and not working.

However, under asymmetric information where the firm does not know worker types, these contracts no longer work, because high-type workers can actually be better off by being employed on the low-type contract (see indifference curve u_H^L). This causes a problem for the firm because their profit is decreased by the high-type workers intentionally being unproductive.

Intuitively, if high-type workers not getting properly compensated is causing the "profit loss", maybe we can solve the problem by paying them to be "optimally productive".

This idea turns out to be pretty fantastic, and the fruition of it is the **screening contracts**. Let's first understand the intuition of how this could be done.

Figure 5.4.1: First-Best Contracts and the Problem Under Information Asymmetry



[20] You can obtain these by solving the firm's profit-maximization problem for each type separately. Make sure to use the fact that all workers, in this case, must have 0 utility.

5.4.2 Imperfect Information and Screening Contracts (Intuition)

Figure 5.4.2 illustrates how screening contracts would work with the following steps:
Step 1: To incentivize high-type to work at level e_H, the firm must pay high-type workers more in the screening contract, call this increase Δ^+.

Step 2: Once the new wage w_H^{MS} is decided, the low-type contract (w_L^{MS}, e_L^{MS}) must lie on the same high-type indifference curve (u_H^{MS}) as the (w_H^{MS}, e_H^{MS}). This is to ensure that *high-type workers must not profitably deviate to the low-type contract*[21].

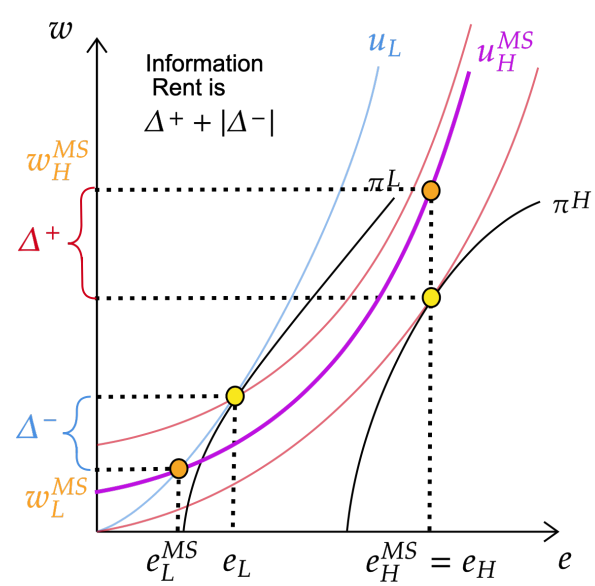
Step 3: To make sure that low-type worker *does not simply not work under this new contract*[22] and to make sure that the firm is profit-maximizing, the screening low-type contract must also be on the indifference curve of low-type simply doing nothing (u_L). As such, the screening low-type contract must be where u_H^{MS} intersects u_L.

Step 4: Naturally, if the original (w_L, e_L) is attractive for high-type workers, the screening low-type contract (w_L^{MS}, e_L^{MS}) must be less attractive, meaning it must have either more work for the same pay, or less work and much lesser pay. Since low-type workers must be indifferent between the screening and first-best contracts and their indifference curve is concave, the screening contract must be of less work and much lesser pay. We will call this difference in pay Δ^-.

Step 5: What the high-type workers gained in aggregate minus what the low-type workers lost in aggregate is called the information rent $\Delta^+ + \Delta^-$.

As such, we have derived (with graphs and logic) the general characterization of monopolistic screening contracts. Let's do this formally with math.

Figure 5.4.2: Screening Contracts Intuition



[21] This restriction is formally called *Incentive Compatibility for High-Type*.
[22] This restriction is formally called *Individual Rationality for Low-Type*.

5.4.3 Firm's Problem Under Asymmetric Information

If the firm does not observe the worker's type θ and wants to screen the worker, i.e., incentivize workers of different types to simply choose different contracts, the following constraints (**Individual Rationality (IR)** and **Incentive Compatibility (IC)**) must hold given screening contracts (w_H, e_H) and w_L, e_L:

- (IR_H) $w_H - c(e_H, \theta_H) \geq 0$
- (IR_L) $w_L - c(e_L, \theta_L) \geq 0$
- (IC_H) $w_H - c(e_H, \theta_H) \geq w_L - c(e_L, \theta_H)$
- (IC_L) $w_L - c(e_L, \theta_L) \geq w_H - c(e_H, \theta_L)$

If you followed the intuitive steps on the last page, it should come as no surprise to you that only IC_H and IR_L are binding constraints. To lazily prove it, see

Lemma 1. IC_H and IR_L imply IR_H: By assumption of $c_\theta < 0$, we have

$$w_H - c(e_H, \theta_H) \geq w_L - c(e_L, \theta_H) \geq w_L - c(e_L, \theta_L) \geq 0$$

Lemma 2. IR_L always binds ($w_L - c(e_L, \theta_L) = 0$): If not, firms can reduce wage until it does, meaning w_L was not profit-maximizing in the first place.

Lemma 3. IC_H always binds ($w_H - c(e_H, \theta_H) = w_L - c(e_L, \theta_H)$): If not, firms can reduce wage until it does, meaning w_H was not profit-maximizing in the first place.

Lemma 4. If IC_H binds and $e_H \geq e_L$, then IC_L is automatically satisfied: This must be true due to single crossing $c_{e\theta} < 0$

With IC_H and IR_L binding, we can now solve the firm's profit-maximization problem:

$$\max_{e_H, e_L, w_H} \lambda \left[\pi(e_H) - c(e_L, \theta_L) - c(e_H, \theta_H) + c(e_L, \theta_H)\right] + (1 - \lambda) \left[\pi(e_L) - c(e_L, \theta_L)\right]$$

Using F.O.C.s to solve, we get

$$[\hat{e}_H] : \pi'(\hat{e}_H) - c_e(\hat{e}_H, \theta_H) = 0$$

$$[\hat{e}_L] : \pi'(\hat{e}_L) - c_e(\hat{e}_L, \theta_L) = \frac{\lambda}{1 - \lambda} \left[c_e(\hat{e}_L, \theta_L) - c_e(\hat{e}_L, \theta_H)\right]$$

Since IC_H is binding, we can compare the optimal high-type wage w_H to the screening high-type wage $\hat{w}_H$:

$$w_H = c(e_H, \theta_H)$$

$$\hat{w}_H = c(\underbrace{\hat{e}_H}_{=e_H}, \theta_H) + \underbrace{\overbrace{c(\hat{e}_L, \theta_L)}^{\Delta^+} - \overbrace{c(\hat{e}_L, \theta_H)}^{\approx\Delta^-}}_{\text{Information Rent}}$$

one must notice a specific caveat here. If information rent is paid through extracting from low type (and the firm suffers a small loss instead of a big loss in profit), there must be some $\lambda = P(\theta = \theta_H)$ such that having separate screening contracts is too costly for the firm.

If λ is close to 1, then there simply are not enough low-type workers to be extracted from, so most of the information rent for high-type workers becomes mostly for the firm to bear. In this case, the firm may simply profit maximize by offering only the high-type first-best contract (w_H, e_H) and only capture the profit $\lambda[\pi(e_H) - w_H]$.

Similarly, if λ is close to 0, then it might be very very costly to get the high-type worker to separate. In this case, the firm may profit maximize by only offering the low-type first-best contract (w_L, e_L) since high-type workers would take that anyways.

All of these sound good, but what if the firm cannot directly observe the offer, or what if the relationship between productivity and effort levels is not deterministic? Would the firm still be able to "screen" for high/low-type workers? Or would the firm simply screen for high/low productivity?

To answer this question, we must make a few adjustments to our model. The end result is commonly called the **Principal-Agent problem**.

5.5 Principal-Agent Problems (Hidden Actions)

Consider the following environment:

- A firm hires a worker who has reservation utility $\bar{u}$.
- To produce output, the worker must exert low effort e_L or high effort $e_H > e_L$, but the chosen effort e is the worker's private information ("hidden action") and the firm cannot observe it (unlike the monopolistic screening).

- The output is publicly observable, but it is an imperfect signal of the true effort. Output $\pi \in [\underline{\pi}, \bar{\pi}]$ stochastically depends on effort e. The conditional CDF of π is $F(\pi \mid e)$ with associated PDF $f(\pi \mid e)$.
- We assume that $F(\pi \mid e_H)$ *strictly* **First-Order Stochastically Dominates** $F(\pi \mid e_L)$ so that $\forall \pi \in [\underline{\pi}, \bar{\pi}]$, $F(\pi \mid e_H) < F(\pi \mid e_L)$. Consequently $E[\pi \mid e_H] > E[\pi \mid e_L]$.
- Firm can offer any contract $w(\pi) : [\underline{\pi}, \bar{\pi}] \to \mathbb{R}$ that depends on the realized output π.
- Worker's outside option is $\bar{u}$
- Worker's payoff is $u(w, e) = v(w) - c(e)$, where $v' > 0$, $v'' \leq 0$, and $c(e_H) > c(e_L)$ (Here we assume that workers are weakly risk-averse so our results are more general. For some cases, we will change this to strictly risk-averse.).
- Firm's payoff is $\pi - w(\pi)$, we assume that the firm is risk neutral.

The important thing to remember is that if efforts are observable, you pay for the effort and not the output and you can perfectly extract worker surplus. If efforts are unobservable, you want to pay more to high output to incentivize workers not to "cheat". In this case, you still want to extract worker surplus completely, but that is only doable to output from the lowest effort.

5.5.1 Base Case: Observable Efforts

Like in our discussion in monopolistic screening, let's see what the equilibrium is when efforts are observable, and see how we can modify from there. This is parallel to the first-best contracts in screening.

Given the stochastic processes of $\pi \mid e_H$ and $\pi \mid e_L$, the firm's profit maximization problem[23] is:

$$\max_{e \in \{e_L, e_H\},\, w(\pi)} \int [\pi - w(\pi)] f(\pi \mid e) d\pi$$
$$s.t. \int v(w(\pi)) f(\pi \mid e)\, d\pi \, - \, c(e) \geq \bar{u} \text{ (Individual Rationality)}$$

Since the firm is solving this problem over 2 spaces e and w, this can be done in a two-step approach:

[23]Notice that in the profit maximization process, the firm must also decide which efforts they should seek, as getting high effort workers might be too costly if the conditional distributions of π given the two effort levels are similar.

Step 1: Given effort level e, what is the best wage schedule? The maximization part can be simplified as:

$$\max_{w(\pi)} \int [\pi - w(\pi)] f(\pi \mid e)\, d\pi \equiv \max_{w(\pi)} \Bigg\{ \underbrace{\int [\pi] f(\pi \mid e)\, d\pi}_{\substack{\text{Some constant} \\ \text{unrelated to } w(\pi)}} - \underbrace{\int [w(\pi)] f(\pi \mid e)\, d\pi}_{\substack{\text{We just need to} \\ \text{minimize this part}}} \Bigg\}$$

Let the IR constraint be binding (because the firm wants to profit maximize) so that $\int v(w(\pi)) f(\pi \mid e)\, d\pi - c(e) = \bar{u}$. The maximization problem is thus simplified to the Lagrangian:

$$\mathcal{L} = -\int w(\pi) f(\pi \mid e) d\pi - \gamma \left(\bar{u} - \int v(w(\pi)) f(\pi \mid e)\, d\pi - c(e) \right)$$

Using point-wise optimization for $\mathcal{L}$, we get the F.O.C.:

$$\frac{\partial \mathcal{L}}{\partial w(\pi)} = -f(\pi \mid e) + \gamma \cdot v'(w(\pi)) \cdot f(\pi \mid e) = 0 \ \Rightarrow \ v'(w(\pi)) = \frac{1}{\gamma}, \forall \pi$$

If $v'' < 0$ (i.e., worker is *strictly risk-averse*), then there is a unique w^* such that $v'(w^*) = \frac{1}{\gamma}$. This means that $\forall \pi \in [\underline{\pi}, \bar{\pi}]$, $w(\pi) = w^*$. Putting this back into IR so that the wage properly rewards effort. Since $v(w)$ is assumed to be strictly increasing, an inverse function $v^{-1}(u)$ exists such that:

$$w^*_{e_H} = v^{-1}(\bar{u} + c(e_H))$$
$$w^*_{e_L} = v^{-1}(\bar{u} + c(e_L))$$

Since we assume that $c_\theta < 0$, $c(e_H) < c(e_L)$, so then it must be that $\bar{u} + c(e_H) < \bar{u} + c(e_L)$. Since v is strictly increasing, we must have

$$w^*_{e_H} = v^{-1}(\bar{u} + c(e_H)) \ > \ v^{-1}(\bar{u} + c(e_L)) = w^*_{e_L}$$

Remark: This means that when efforts are observable (but still are only stochastically correlated with productivity) the optimal wage is a flat wage floor for e_L and add on "rewards" for $e - e_L$.

Step 2: Should the firm pick e_H or e_L?

This question depends on the exact form of the utility function and the conditional distribution of π given effort. In general, the firm solves:

$$\underset{e\in\{e_H,e_L\}}{argmax}\left\{\int \pi f(\pi \mid e_H)\, d\pi - v^{-1}(\bar{u}+c(e_H))\ ,\ \int \pi f(\pi \mid e_L)\, d\pi - v^{-1}(\bar{u}+c(e_L))\right\}$$

5.5.2 Case of Interests: Unobservable Efforts for a Risk-Averse Worker

We've shown that the maximization problem is simplified to a wage-per-dollar-profit minimization problem. But since we cannot observe efforts, we want our wage to incentivize the effort level we want (IC). For given effort e, we have

$$\min_{w(\pi)} \int w(\pi) f(\pi \mid e) d\pi$$
$$s.t. \begin{cases} \int v(w(\pi)) f(\pi \mid e)\, d\pi \ -\ c(e) \geq \bar{u} & (IR) \\ e \in \underset{\tilde{e}}{argmax}\ \int v(w(\pi)) f(\pi \mid \tilde{e})\, d\pi \ -\ c(\tilde{e}) & (IC) \end{cases}$$

Like before, we have a two-step process:

Step 1: Suppose that we want to incentivize some $e_H > e_L$, then IR_H and IC_H must hold, meaning we have the constraints:

$$\begin{cases} \int v(w(\pi)) f(\pi \mid e_H)\, d\pi \ -\ c(e_H) \geq \bar{u} & (IR_H) \\ \int v(w(\pi)) f(\pi \mid \tilde{e_H})\, d\pi \ -\ c(\tilde{e_H}) \geq \int v(w(\pi)) f(\pi \mid \tilde{e_L})\, d\pi \ -\ c(\tilde{e_L}) & (IC_H) \end{cases}$$

This means that our Lagrangian for this point-wise minimization problem is:

$$\mathcal{L} = \int w(\pi) f(\pi \mid e) d\pi + \gamma\left[\bar{u} - \int v(w(\pi)) f(\pi \mid e_H)\, d\pi + c(e_H)\right]$$
$$+ \mu\left[\int v(w(\pi)) f(\pi \mid e_L)\, d\pi \ -\ c(e_L) - \int v(w(\pi)) f(\pi \mid e_H)\, d\pi \ +\ c(e_H)\right]$$

The F.O.C. is hence:

$$\frac{\partial \mathcal{L}}{\partial w(\pi)} = f(\pi \mid e_H) - \gamma\Big[v'(w(\pi)) f(\pi \mid e_H)\Big]$$
$$- \mu\Big[v'(w(\pi)) f(\pi \mid e_L) - v'(w(\pi)) f(\pi \mid e_H)\Big] = 0$$

$$\text{(Divide by } f(\pi \mid e_H)) \quad \Rightarrow \ 1 = \gamma \cdot v'(w(\pi)) + \mu \cdot v'(w(\pi)) \left[1 - \frac{f(\pi \mid e_L)}{f(\pi \mid e_H)}\right]$$

Solving for $v'(w(\pi))$, we get

$$v'(w(\pi)) = \frac{1}{\gamma + \mu\left[1 - \frac{f(\pi|e_L)}{f(\pi|e_H)}\right]} \quad (\star)$$

If $\mu = 0$, then $v'(w(\pi)) = \frac{1}{\gamma}$, which is the same as the observable case, but that is not possible, so $\mu > 0$ and IC_H must be binding.

If $\gamma = 0$, then $v'(w(\pi)) = \left[\mu - \mu\frac{f(\pi|e_L)}{f(\pi|e_H)}\right]^{-1}$. Since $v(w(\pi))$ is assumed to be strictly increasing, this must mean that $f(\pi \mid e_L) < f(\pi \mid e_H)$, $\forall\pi$.

If that is the case, $F(\pi \mid e_L) = \int_{\underline{\pi}}^{t} f(\pi \mid e_L)\, d\pi < \int_{\underline{\pi}}^{t} f(\pi \mid e_H)\, d\pi = F(\pi \mid e_H)$. But we assumed that $F(\pi \mid e_H)$ FOSD $F(\pi \mid e_L)$ with strict inequality on the interior of *supp*(π), which is the exact opposite. By contradiction, $\gamma > 0$, so IR_H is also binding.

Step 2: For the low effort e_L, the firm must offer a low wage that disincentivizes the worker to work low. Since IR_L must bind, the low page must be the same as the observable case $w(\pi) = w_{e_L}$.

In summary, we have found that the profit-maximizing wage schedules are such that:

- Observable Efforts: $v(w_{e_H}) = \bar{u} + c(e_H)$
- Unobservable Efforts: $E\left[v(w_{e_H}(\pi)) \mid e_H\right] = \bar{u} + c(e_H)$

Since $v(w(\pi))$ is assumed to be strictly concave and strictly increasing, by Jensen's inequality, $E\left[w_{e_H}(\pi) \mid e_H\right] > w_{e_H}$. So the firm will have to pay more to incentivize higher effort in the unobservable case.

Intuitively, this result makes a lot of sense. If the risk-averse worker's effort does not guarantee high productivity (which the firm pays for), they need higher compensation for when they reach the "target" productivity to offset the loss for when they reach "low" productivity even if they put in high effort.

If, on the other hand, that workers are risk-neutral ($u(\cdot)$ is linear), then the μ in equation $(\star)$ must be 0. As such, in the unobservable efforts case with risk-neutral workers, we have

$$E\left[v(w_{e_H}(\pi)) \mid e_H\right] = v(E[w_{e_H}(\pi) \mid e_H)]) = \bar{u} + c(e_H)$$

and so the contracts would be the same as the observable case **in the effort the firm wants to implement**.

Now, let's focus back on equation ($\star$). Assuming that the distribution of the stochastic process is known, $f(\pi \mid e_L)$ and $f(\pi \mid e_H)$ are the likelihood function that maps productivity to effort levels. Let the observable efforts wage be w^{**} such that $v(w^{**}) = \frac{1}{\gamma}$, we can thus see:

$$\frac{f(\pi \mid e_L)}{f(\pi \mid e_H)} < 1 \iff w(\pi) > w^{**} \tag{1}$$
$$\frac{f(\pi \mid e_L)}{f(\pi \mid e_H)} = 1 \iff w(\pi) = w^{**} \tag{2}$$
$$\frac{f(\pi \mid e_L)}{f(\pi \mid e_H)} > 1 \iff w(\pi) < w^{**} \tag{3}$$

Meaning that the wage the firm will pay depends on the likelihood ratio of the two distributions. In plain words, if the productivity shown makes the firm think the worker is more likely to have put in e_H (equation (1)), then the firm will pay the higher wage. If the productivity shown makes the firm think the worker is more likely to have put in e_L (equation (3)), then the firm will pay a lower wage. If the productivity shown makes the firm think the worker is equally likely to have put in either e_H or e_L, then the firm will pay the flat wage.

What this means is that since likelihood ratios between distributions on the same support are not necessarily monotonic, wage schedules are generally not monotonic in productivity either. On the other hand, if the likelihood ratio is monotonic on the support, then the wage schedule will be monotonic. If the likelihood ratio is monotonic on the support, we say that it satisfies the **Monotone Likelihood Ratio Property (MLRP)**.

We have discussed several solutions to the information asymmetry problem, but you probably have noticed that all of these solutions are less efficient than the "first-best" solutions (under perfect information). As such, all we have studied is how to make a compromise (and accept dead-weight-loss in equilibrium) such that the market will not completely unravel.

More importantly, the solutions we have discussed distribute the loss to either the buyer (firms in screening and hidden action) or the seller (workers in signalling) but never both. As aspiring social scientists, we must explore the possibility of games that can distribute the loss to all agents in equilibrium. This is what motivates the study of **Mechanism Design**, and we shall take a brief tour through that world.

5.6 Mechanism Design

Recall that at the beginning of our discussion of Information Asymmetry, we talked about how Mechanism design is *the design of a game by the less informed party to elicit both information and actions from the more informed parties.* Among actions that can be taken by the less-informed party, *Monopolistic Screening* is where the firm designs a contract such that agents reveal their types and *Principal-Agent Problems* is where the firm designs a contract such that regardless of type, the firm just wants to see a specific action. Typically, this results in an "undesired" action getting the lowest possible payoff.

In other words, screening is about eliciting types, and hidden actions is about eliciting actions. One must thus naturally ask, if it possible to do both? Is it possible to design a contract such that types are revealed while the firms' desired outcome is achieved?

Think about the difference between screening and hidden actions. The monopolistic firms "maximize profit" in both cases, but are they the same profit? Or is there something more nuanced? In our brief discussion of mechanism design, we will attempt to take a "big-picture" approach into answering these questions.

Consider the following environment:

- I agents $i \in \mathcal{I} = \{1, \dots, I\}$ who each have private information about themselves.
- Let X be the of outcomes/allocations.
- Let ΔX be the set of lotteries over outcomes/allocations.
- Let Θ_i be the set of possible payoff types for agent $i \in \mathcal{I}$
  - $\theta_i \in \Theta_i$ is private information
  - Denote the set of payoff profiles as $\Theta \equiv \prod_{i=1}^{\mathcal{I}} \Theta_i$
  - The payoff profile $\theta = (\theta_1, \dots, \theta_I)$ follows a random process with joint CDF $F(\theta)$
- Let $u_i(x, \theta_i)$ be the payoff for type θ_i agent when x is the outcome.

Perhaps it is best to think about mechanism design from a Social Planner's perspective as the easiest application of this big-picture approach is to maximize overall welfare[24]. To do

[24] Keep in mind that maximizing overall welfare need not be the case in specific problems. Think about how a social planner can heterogeneously weigh each agent's welfare.

so, we must first define the objective function.

Definition (SCF): A **Social Choice Function** is a function f that maps the set of type profiles to the set of outcomes/allocations:

$$f : \Theta \to X$$

Definition (Allocation): An allocation $x \in X$ is a vector that specifies the outcome and costs for each individual. For $\mathcal{I}$, we have

$$x = (y_1, \ldots, y_I, t_1, \ldots, t_I)$$

An outcome is said to be **feasible** if $\sum_{\mathcal{I}} y_i \leq$ Resource Constraint

Definition (Efficiency): An SCF is said to be **Ex-Post Efficient** if $\nexists(\theta, x')$ such that for some $i \in \mathcal{I}$ and $\forall j \in \mathcal{I} \setminus \{i\}$

$$[u_i(x', \theta_i) > u_i(f(\theta), \theta)] \;\wedge\; [u_j(x', \theta_j) \geq u_i(f(\theta), \theta)]$$

In other words, an SCF is ex-post efficient if and only if $\forall \theta \in \Theta$, $f(\theta) \in X$ is Pareto Optimal.

Definition (Mechanism): A mechanism is a game $\Gamma = \{S_1, \ldots, S_I, g(\cdot)\}$ that consists of *a collection of strategy profiles* and *an outcome function* $g : S \to X$.

Philosophically, a mechanism is no different from an SCF, but mathematically, it is much easier to work with. To design an *efficient* SCF, one must know the realization of type, and then maximize based on those. But there can be arbitrarily many type combinations, which makes implementing the ex-post efficient SCF practically impossible in most cases.

Using a mechanism simplifies the problem. If we assume that agents will not play any dominated strategies given their type θ_i, then we can reduce the mapping to a subset of X. In that case, we only need to find mutual best responses (NEs) and make sure the mechanism picks the NE, for each $\theta \in \Theta$ such that the corresponding SCF is ex-post efficient.

Example: Solving Auctions - SCF vs. Mechanism

Social Choice Function:
In single indivisible-good auctions for I agents, the SCF is a mapping from player valuations V to the outcome[a] of the auction

$$(0, \ldots, 0, v_i, 0, \ldots, 0, \pi, t_1, \ldots, t_i = b_i, \ldots, t_I, t^F = 0)$$

where t_j represents the costs to each player in this auction, π represents the firm's profit, and t^F represents the firm's reservation value (set to 0 for simplicity).

Intuitively, the *candidates* of our ex-post efficient SCF f must satisfy:

- A player pays at most up to their valuation if they win
- The firm gets at least profit 0 ($\pi - t^F \geq 0$)

Using the definition of ex-post efficiency, we need to further refine to either

- A firm must maximize their profit given the social choice function (implying player with the highest valuation wins and pays exactly the valuation);
  Or
- The player with the highest valuation wins and pays the lowest amount acceptable to the firm

As such, the ex-post efficient SCF must satisfy either (Assume $v_k = \max\limits_{i \in \mathcal{I}}\{v_i\}$) :

$$f(\theta) = (0, \ldots, 0, v_k, \ldots, 0, \pi = v_k, t_1 = 0, \ldots, t_{k-1} = 0, t_k = v_k, t_{k+1} = 0, \ldots t_I = 0, t^F = 0)$$

or

$$f(\theta) = (0, \ldots, 0, v_k, \ldots, 0, \pi = 0, t_1 = 0, \ldots, t_{k-1} = 0, t_k = 0, t_{k+1} = 0, \ldots t_I = 0, t^F = 0)$$

The former is what we are used to seeing in auction problems, the latter may seem counter-intuitive but is perfectly in line with the definition of ex-post efficiency. In fact, certain "combinations" of the two can be ex-post efficient (e.g., second-price auctions).

Hopefully, this above example of SCF makes you realize just how difficult working with SCF actually is, given all kinds of types and payment schemes. Now consider this same problem but with a mechanism.

Mechanism:
A Mechanism in this case is $\Gamma = \{b_1, \ldots, b_I, g(\cdot)\}$ such that:

$$g(b_1, \ldots, b_i) = (y_1, \ldots, y_I, \pi, t_1, \ldots, t_I, t^F = 0)$$

Using results derived in auction theory, we know that the BNE is generally where the agent with the highest valuation wins the auctions and firms can profit maximize by setting a reservation price (In this case, we shall set it to 0)[b].

Notice how this objective coincides with the SCF case, but because we know how to solve an auction problem, we can systematically eliminate $g(\cdot)$ that does not lead to a BNE that satisfy (e.g., we can easily rule out an auction where the lowest bid wins the auction, as the BNE of that game would be everyone bid 0).

[a]The written outcome here is the outcome where player i wins the auction.
[b]Note that in auction theory, we never said that the auction firm has actually charge the winner a price. If their reservation price is 0, then charging 0 on the winning bid can still lead to the same BNE, as long as players don't know that ahead of time (this gives us the flexibility of treating mechanisms and SCFs as equivalent objectives).

Assuming I have convinced you that designing a mechanism is, in spirit, the same as finding an ex-post efficient social choice function, but much simpler. To formalize a way to find the "best" mechanism that matches to an SCF, we must take a somewhat axiomatic approach.

Definition (DRM): A **Direct Revelation Mechanism** is a game $\Gamma = \{S_1, \ldots, S_I, g(\cdot)\}$ such that $s_i \in \underset{s \in S_i}{argmax}\ u(s_i, s_{-i} \mid \theta_i)$ and the mapping $s : \theta_i \mapsto s_i$ is 1-1 mapping $\forall i$.
We can thus rewrite a DRM with types instead of strategies,

$$\Gamma = \{S_1, \ldots, S_I, g(\cdot)\} \equiv \Gamma^D = \{\Theta_1, \ldots, \Theta_I, g(\cdot)\}$$

In plain English, a DRM is a mechanism where every agent's Nash equilibrium strategy *directly reveal* their type.
Think about this like in a separating equilibrium where high-type workers' equilibrium strategy is to reveal themselves as high-types and low-type workers' equilibrium strategy is to reveal themselves as low-types.

Definition (DSE): A strategy profile $s = (s_1, \dots, s_I)$ is a **Dominant Strategy Equilibrium** of the mechanism $\Gamma = \{S_1, \dots, S_I, g(\cdot)\}$ if $\forall s' \neq s \in S$

$$u_i\left(g(s_i(\theta_i), s_{-i}) \mid \theta_i\right) \geq u_i\left(g(s_i'(\theta_i), s_{-i}) \mid \theta_i\right)$$

In plain English, this a strategy profile is a *DSE* if it is a Nash Equilibrium given agents' types θ (Note that since it is given types, the NE constitutes only either weakly or strictly dominant strategies).

Definition (DSI): We say that a mechanism $\Gamma = \{S_1, \dots, S_I, g(\cdot)\}$ **Implements** the SCF $f(\cdot)$ in **Dominant Strategies** if there exists a *dominant strategy equilibrium* s^* such that $g(s^*(\theta)) = f(\theta)$.

In plain English, *DSI* describes when designing a mechanism is equivalent to finding the ex-post efficient SCF.

But how do we know that the SCF we want to find is actually "solvable" through a mechanism. In other words, we know when we can map a mechanism to an SCF (like injection, but not limiting to 1-1), but how do we know, for some SCFs, that there are mechanisms that map to it (like surjection). To answer that, we must introduce **The Revelation Principle for Dominant Strategies**.

In principle, we must consider all possible mechanisms when trying to figure out if there is such mapping. Luckily, it turns out (with proof omitted) that it suffices to ask whether a particular SCF is **truthfully implementable in dominant strategies**.

Definition (TIDS): An SCF $f(\cdot)$ is said to be **Truthfully Implementable in Dominant Strategies** if a *Direct Revelation Mechanism* implements $f(\cdot)$ using a *Dominant Strategy Equilibrium*. Essentially, an SCF is truthfully implementable in dominant strategies if we can map a DRM whose NE yields the outcome equivalent to the SCF.

Proposition: The Revelation Principle for Dominant Strategies
Suppose there exists a mechanism $\Gamma = \{S_1, \dots, S_I, g(\cdot)\}$ that implements the SCF $f(\cdot)$ in dominant strategies. Then, $f(\cdot)$ is truthfully implementable in dominant strategies by some DRM $\Gamma^D = \{\Theta_1, \dots, \Theta_I, g(\cdot)\}$ and its DSE s^*.

Proof: The Revelation Principle for Dominant Strategies

Suppose $\Gamma = \{S_1, \dots, S_I, g(\cdot)\}$ implements f in dominant strategies, then there exists a strategy profile s^* such that

$$g(s^*(\theta)) = f(\theta), \forall\theta$$

and

$$u_i\left(g(s_i^*(\theta_i), s_{-i}) \mid \theta_i\right) \geq u_i\left(g(s_i'(\theta_i), s_{-i}) \mid \theta_i\right), \forall s' \neq s \tag{1}$$

If agents are not truthful (e.g., agent i uses θ_j's strategy instead), and use the alternative strategies $s_i'(\theta_i) = s_i^*(\theta_j)$ in equation (1), then we get

$$u_i\left(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}) \mid \theta_i\right) \geq u_i\left(g(s_i^*(\theta_j), s_{-i}^*(\theta_{-i}) \mid \theta_i\right) \tag{2}$$

Since Γ implements f in dominant strategies, this must mean equation (2) can be rewritten as

$$\underbrace{u_i(f(\theta_i, \theta_{-i} \mid \theta_i)}_{\substack{\text{Payoff of agent } i \text{ if their} \\ \text{strategy is true to their type}}} \quad \geq \quad \underbrace{u_i(f(\theta_j, \theta_{-i} \mid \theta_i)}_{\substack{\text{Payoff of agent } i \text{ if their} \\ \text{strategy is NOT true to their type}}} \tag{3}$$

Equation (3) then tells us that being truthful will dominate being deceitful, so if Γ implements f in dominant strategies, then f is truthfully implementable. □

This result seems nice and clean, but the fact of the matter is that not all SCFs are implementable in dominant strategies. Consider the Bibilical story of King Solomon's problem as an example of a non-truthfully implementable SCF:

Example: King Solomon's Problem

Two women claim to be the mother of a child. One is the mother (M), and the other is an imposter (I). The king wants to give the child to the real mother (i.e., giving the child to the actual mother must be part of the ex-post efficient SCF).

King Solomon then designed a mechanism such that if no definitive choice is made, the child is cut in half (denoted as $x = \varnothing$). Otherwise, the child is given to the "mother" (denoted as $x = x_i$ if player i is given the child).

Suppose that the players have the following utility functions:

$$u_i(x_i \mid \theta_i = M) = 2 > u_i(x_j \mid \theta_i = M) = 1 > u_i(\varnothing \mid \theta_i = M) = 0$$
$$u_i(x_i \mid \theta_i = I) = 2 > u_i(\varnothing \mid \theta_i = I) = 1 > u_i(x_j \mid \theta_i = I) = 0$$

Suppose that each player's strategy set is $s_i = \{M, I\}$, so the payoff table is

P_1 \ P_2		$\theta_2 = M$		$\theta_2 = I$	
		M	I	M	I
$\theta_1 = M$	M			$0, 1$	$\underline{2}, 0$
	I			$\underline{1}, \underline{2}$	$0, \underline{1}$
$\theta_1 = I$	M	$\underline{1}, 0$	$\underline{2}, \underline{1}$		
	I	$0, \underline{2}$	$1, 0$		

The unique NE in this game is for the imposter to always claim to be the mother, and for the mother to claim to be an imposter.

Since the NE in this game is "unique" given realization of types, King Solomon's SCF where the actual mother is given the child is **NOT** implementable by the mechanism and hence it is **not** truthfully implementable in dominant strategies by this mechanism.

In reality, King Solomon knows this, and changed the mechanism ex-post, and gave the child to the woman who claims to be the imposter. One might think that the "new" mechanism is implementable, as the strategies are still one-to-one mappings between actual type and the strategy. But this is incorrect, since changing the mechanism ex-ante would be equivalent to re-labelling the strategies, and the mechanism itself still does not truthfully implement King Solomon's SCF.