Michigan State EC813B Organized Lecture Notes

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This is a special thank-you to Younghee Son, who provided me with meticulous notes from all the lectures where I didn't know what was going on. Her explanations in her own notes have so much explanatory power that if we regress the entirety of "what we were supposed to learn in this class" on her notes we would get a perfect fit[1]*. It looks nicer if I have a full paragraph, so here is lorem ipsum.*

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[1] That being said, if something is wrong in my organized notes, it is a me problem. Younghee is not responsible for any faults in this document. Please let me know if you come across any mistakes/typos in here so I can update it.
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1 Static Representative Agent Problems

In Macroeconomics, we want to build on the work of consumer demand and the general equilibrium framework to try to make predictions about the "larger" economy as opposed to focusing on individual markets. To do so, we need to first revisit the *basic representative agent problem* to see how we can scale our results from the study of microeconomics.

First we need to define our environment. For now, we will work in an economy that has:

- N identical consumers with the utility function $u(c, l)$ where c denotes consumption and l denotes leisure. We will assume that:
  - $u(c, l)$ is strictly increasing in both arguments, strictly concave, and is twice differentiable
  - $\lim_{c \to 0} u_c(c, l)|_{l>0} = \lim_{l \to 0} u_l(c, l)|_{c>0} = \infty$[2]
- M identical firms with production technology $y = zf(k, n)$ where z denotes total factor productivity (TFP), k denotes capital, and n denotes labor. We will assume that:
  - $f(k, n)$ is strictly increasing in both arguments, strictly quasi-concave, and is twice differentiable, and homogeneous of degree 1 $f(\lambda k, \lambda n) = \lambda f(k, n)$).
    Note that this means if we differentiate f with respect to λ and evaluate at $\lambda = 1$ we have

$$f_k \cdot k + f_n \cdot n = f(k, n)$$

  - $\lim_{k \to 0} f_k(k, n)|_{n>0} = \lim_{n \to 0} u_n(k, n)|_{k>0} = \infty$
- Endowments: Every consumer is endowed with $\frac{k_0}{N}$ units of capital and 1 unit of time to allocate between l and n^s
- 3 Markets: Goods market (price $p = 1$), Labor market (price w), and Capital market (price r being the rental rate per period)

[2] I am well-aware that $= \infty$ is not a thing, Patrick. Go tell that to Andrei.

1.1 The Walrasian Problem

With the environment defined, we can first look at things from the Walrasian perspective. In the Walrasian framework, consumers solve the problem

$$\max_{c,l} u(c,l) \textit{ subject to} \begin{cases} 0 \leq c \leq w(1-l) + r \cdot k^s \\ 0 \leq l \leq 1 \\ 0 \leq k^s \leq \frac{k_0}{N} \\ k^s = \frac{k_0}{N} \end{cases}$$

where the last constraint comes from assuming non-negative rental rate r. This problem can be solved by the Lagrangian

$$\mathcal{L} \equiv u(c,l) - \lambda[c - w(1-l) - r\frac{k_0}{N}]$$

Solving the first order conditions, we get the optimizing condition

$$\frac{u_c}{u_l} = \frac{1}{w} = \frac{\text{Marginal Cost of Consumption}}{\text{Marginal Cost of Leisure}} = \frac{\text{Price of Good}}{\text{Opportunity Cost of Not Working}}$$

From this condition, we can calculate leisure demand as a function of wage w and rental rate r since

$$u_c(w(1-l) + rk_0, l) \cdot w = u_l(w(1-l) + rk_0, l)$$

On the other side of the market, we have the typical firm's profit-maximization problem:

$$\max_{k,n} zf(k,n) - wn - rk$$

Notice that this is presented as an ex-ante maximization problem and not an ex-post cost minimization problem since we eventually want to look at steady-state solutions where agents have the ability to plan long-term.

Solving the firm's problem as an unconditional maximization problem, we get the first order conditions:

$$zf_k(k,n) = r$$
$$zf_n(k,n) = w$$

Recall that, by assumption, we have $f(k,n) = f_k k + f_n n$ meaning that, at optimal, the firm's profit can be represented as:

$$zf(k,n) - wn - rk = zf(k,n) - zf_n n - zf_k k = 0$$

This means that at optimal, price-taker firms should be making 0 economic profit (meaning positive nominal profit).

We can now characterize the competitive equilibria in this economy as a list (c,l,n,f) with prices (w,r) such that

(i) Representative consumers maximize their utilities

(ii) Representative firms maximize their profits

(iii) Every individual market clears[3]

Notice that we have

- 5 equations: $u_l(c,l) = w \cdot u_c(c,l)$, $zf_n = w$, $zf_k = r$, $n = 1-l$, $k = k_0$
- 5 unknowns: l, n, k, w, r

Through some substitutions, we get the equation:

$$u_l(zf_n(1-l) + zf_k k_0, l) - zf_n \cdot u_c(c,l) = 0$$

Meaning that we can solve for these variables

$$l^* \Rightarrow n^* \Rightarrow y^* \Rightarrow c^* \Rightarrow w^*, r^*$$

1.2 The Social Planner's Problem (Pareto Optima)

The social planner's problem is one without prices. Instead of finding equilibria with prices, the Pareto approach focuses on indifference and marginal rates of substitution[4]. Assuming that representative firms do not want to waste their products, we can solve for the Pareto optima with

$$\max_l u(c,l) \ s.t. \ c = zf(k_0, 1-l)$$

[3]Readers may recall the characterization of an equilibrium being allocations with non-positive excess demand in a market. Since we are using the Lagrangian to obtain interior solutions, we have assumed away 0-prices and hence the equilibria with negative excess demands.

[4]See J.R.Hicks *Value and Capital* section I for more awesome discussion.

giving us the optimizing condition:

$$u_c \cdot z \underbrace{f_l}_{=-f_n} + u_l = 0$$

Notice that this condition turns out to be identical to what we saw in the Walrasian approach, showing us that the Competitive equilibrium is Pareto optimal, coinciding with the result of the *First Welfare Theorem*.

To see that the *Second Welfare Theorem* also applies, it is perhaps best with a simple example. Consider an economy with firms having the production function $y = z \cdot n$. The two approaches (Walras and Pareto) solves for

$$\text{Walrasian: } \begin{cases} \text{RA problem: } \max_{c,l} u(c,l) \; s.t. \; c = w(1-l) \\ \text{RF problem: } \max_{n} zn - wn \end{cases}$$
$$\text{Pareto: } \max_{c,l} u(c,l) \; s.t. \; c = z(1-l)$$

From these setups, we get the common F.O.C.

$$u_l(z(1-l), l) - zu_c(z(1-l), l) = 0 \tag{1}$$

We can thus examine, in static, what happens if there is a shock to TFP, changing how labor is valued. We can totally differentiate equation (1) with respect to z and get:

$$\begin{aligned} & u_{lc} \cdot (1 - l - zl_z, l) + u_{ll} \cdot l_z - u_c - zu_{cc} \cdot (1 - l - zl_z) - zu_{cl} \cdot l_z = 0 \\ \Rightarrow\ & u_{cl} \cdot (1-l) - u_c - zu_{cc}(1-l) = [2zu_{cl} - u_{ll} - z^2 u_{cc}] \cdot l_z \\ \Rightarrow\ & l_z = \frac{u_{cl} \cdot (1-l) - u_c - zu_{cc}(1-l)}{2zu_{cl} - u_{ll} - z^2 u_{cc}} \end{aligned}$$

where l_z denotes the total "price effect" of changes in TFP on equilibrium level of labor. Recall that price effect here is in the Marshallian sense, which can be decomposed into substitution effect and income effect. The key is that substitution effect (Hicksian price effect) is changes on a fixed indifference curve, without considering income.

To get substitution effect of Δz on l, we need to first assume that at the original level of production, the representative consumers have utility $u(c,l) = h \in \mathbb{R}$. By the implicit function theorem, we can thus differentiate both sides with respect to z and get:

$$u_c \cdot c_z + u_l \cdot l_z = 0 \;\Rightarrow\; c_z = -\underbrace{\frac{u_l}{u_c}}_{=w=z} \cdot l_z = -z l_z \;\Rightarrow\; u_l(c,l) - z u_c(c,l) = 0 \tag{2}$$

Redoing the derivative of equation (1) now that we know how c_z and l_z are related, we get

$$\begin{aligned} u_{lc} \cdot c_z + u_{ll} \cdot l_z - u_c - z u_{cc} c_z - z u_{cl} l_z = 0 \\ \Rightarrow\; -u_{lc} \cdot z \cdot l_z + u_l \cdot l_z - u_c + z^2 u_{cc} l_z - z u_{cl} \cdot l_z = 0 \end{aligned}$$

Solving for l_z, we get the substitution effect l_z^s:

$$l_z^s = \frac{-u_c}{2z u_{cl} - u_{ll} - z^2 u_{cc}}$$

So the total price effect can be decomposed into:

$$l_z = \frac{u_{cl} \cdot (1-l) - u_c - z u_{cc}(1-l)}{2z u_{cl} - u_{ll} - z^2 u_{cc}} = \underbrace{\frac{-u_c}{2z u_{cl} - u_{ll} - z^2 u_{cc}}}_{\text{Substitution Effect}} + \underbrace{\frac{u_{cl} \cdot (1-l) - z u_{cc}(1-l)}{2z u_{cl} - u_{ll} - z^2 u_{cc}}}_{\text{Income Effect}}$$

1.3 Government

Now that we have a general model for consumers and firms, let's throwing government, the painfully visible hand. The inclusion of government means that our economy now has a little bit more elements:

- Let g denote government expenditure
- Let τ denote lump-sum taxes[5]
- The government must balance its budget: $g = \tau$
- The social welfare function is $W(c, l, g) = u(c, l) + v(g)$

Continuing our simple example with $y = z \cdot n$, the new first order condition is:

$$z u_c(z(1-l) - g, l) = u_l(z(1-l) - g, l)$$

We can thus study the effect of government expenditure on leisure, using the same method we studied the effect of z:

[5] Since the τ is lump-sum and not distortionary taxes, the *First Welfare Theorem* will hold.

Totally differentiating the first order condition with respect to g, we get[6]:

$$\begin{aligned} & zu_{cc} \cdot (-zl_g - 1) + zu_{cl} \cdot l_g - u_{lc} \cdot (-zl_g - 1) - u_{ll} \cdot l_g = 0 \\ \Rightarrow\ & l_g[-z^2 u_{cc} + 2zu_{cl} - u_{ll}] = zu_{cc} - u_{cl} \\ \Rightarrow\ & l_g = \frac{z \cdot \overbrace{u_{cl}}^{>0} - \overbrace{u_{cc}}^{<0}}{z^2 \cdot \underbrace{u_{cc}}_{<0} - 2z \underbrace{u_{cl}}_{>0} + \underbrace{u_{ll}}_{<0}} < 0 \end{aligned}$$

So the total effect of an increase on government expenditure decreases leisure and increases labor.

Next, we know that, in equilibrium, we have $c = z(1-l) - g$. Differentiating both sides with respect to g we get

$$\begin{aligned} c_g = -zl_g - 1 &= \frac{z^2 u_{cc} - zu_{cl}}{z^2 u_{cc} - 2zu_{cl} + u_{ll}} - 1 \\ &= \frac{zu_{cl} - u_{ll}}{z^2 u_{cc} - 2zu_{cl} + u_{ll}} \in (-1, 0) \end{aligned}$$

So the total effect of an increase on government expenditure decreases consumption less than proportionally.

We can also work out the effect of g on production using the same method, but based on our calculation for l_g, we know that production would increase.

Armed with this static model, we can now move to a dynamic model. Consider the infinite horizon problem

$$\max_{c_t, l_t} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)\ s.t. \begin{cases} c_t & = w_t(1 - l_t) - \tau_t - s_{t+1} + (1 + r_t)s_t \\ g_t & = \tau_t + b_{t+1} - (1 + r_t)b)t,\ b_0 = 0 \\ y_t & = z_t n_t \end{cases}$$

[6]Since $u(c, l)$ is assumed to be strictly concave and strictly increasing in both arguments, we have $u_{cc}, u_{ll} < 0$ and $u_{cl} > 0$.

The representative agent faces the inter-temporal budget constraint:

$$c_0 + \sum_{t=1}^{\infty} \frac{c_t + \tau_t + s_{t+1}}{\prod_{t=1}^{\infty} 1 + r_t} = w_0(1 - l_0) - \tau_0 + \sum_{t=1}^{\infty} \frac{w_t(1 - l_t) + (1 + r_t)s_t}{\prod_{t=1}^{\infty} 1 + r_t}$$

By the non-Ponzi condition, we have $\lim_{t \to \infty} \frac{s_t}{\prod_{t=1}^{\infty} 1+r_t} = 0$. So the first order conditions are:

$$[c_t] : \beta^t u_c(c_t, l_t) = \frac{\lambda}{\prod_{t=1}^{\infty} 1 + r_t}$$

$$[l_t] : \beta^t u_l(c_t, l_t) = \frac{\lambda w_t}{\prod_{t=1}^{\infty} 1 + r_t}$$

Giving us something awfully familiar:

$$\frac{u_c(c_t, l_t)}{u_l(c_t, l_t)} = w_t, \quad \beta \frac{u_c(c_{t+1}, l_{t+1})}{u_c(c_t, l_t)} = \frac{1}{1 + r_{t+1}}$$

From here, we can define the equilibrium as a list of $(c_t, l_t, n_t, b_t, \tau_t, s_t)_0^\infty$ and $(w)_0^\infty$ such that

(i) Representative consumers maximize $\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$ given $(w_t, \tau_t)_0^\infty$

(ii) Representative firms maximize $\sum_{t=0}^{\infty} \frac{z_t(1 - l_t) - w_t(1 - l_t)}{\prod_{t=1}^{\infty} 1 + r_t}$

(iii) Government budget is balanced: $g_0 + \sum_{t=1}^{\infty} \frac{g_t + (1 + r_t)b_t}{\prod_{t=1}^{\infty} 1 + r_t} = b_0 + \tau_0 + \sum_{t=1}^{\infty} \frac{\tau_t + b_{t+1}}{\prod_{t=1}^{\infty} 1 + r_t}$

(iv) Each individual market clears: $c_t + g_t = y_t$, $1 - l_t = n_t$, $b_t = s_t$

Subbing (iii) into RAIBC, we see that the consumers' budget constraint no longer contains taxes, only government expenditure. This is the idea of **Ricardian Equivalence**.

Definition (Ricardian Equivalence)[7]: The path of consumption $(c_t)_0^\infty$ does not depend on the path of taxation $(\tau_t)_0^\infty$.

[7] Note that for us to get to the conclusion of Ricardian equivalence, we had to assume (a) Complete markets in each period, and (2) No distortionary taxes.

2 Metric Spaces[8]

2.1 Metric Space Fundamentals

Consider the ordered field $(\mathbb{R}, +, \cdot, \geq)$. We need a little more structure to make this a metric space. Specifically, we need to define how distances are measured in this field. Conventionally, we use the following distance functions:

1. (In $\mathbb{R}^1$) $d(x, y) : \mathbb{R} \to \mathbb{R}$, $d(x, y) = |x - y|$
2. (Euclidean Distance in $\mathbb{R}^n$) $d(x, y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, $d(x, y) = \|x - y\| = \sqrt{\sum_i (x_i - y_i)^2}$

Definition (Metric): A **metric** on a set X is a function $d : X \times X \to \mathbb{R}_+$ such that:

(i) **Separation** $d(x, y) = 0 \iff x = y$

(ii) **Symmetry** $d(x, y) = d(y, x)$

(iii) **Triangle Inequality** $\forall x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$

Definition (Metric Space): A **metric space** M on the set X is a *pair* $M \equiv (X, d)$ where $d : X \times X \to \mathbb{R}_+$ is a metric

An example of a non-standard metric is **the discrete metric**. Consider the following distance function (metric):

$$d_0(x, y) = \begin{cases} 1 & , x \neq y \\ 0 & , x = y \end{cases}$$

One can check that the function $d_0(x, y)$ satisfies the 3 criteria for a distance function to be a metric. The following are the more "useful" metrics:

The p-metric: $\forall p \in [1, \infty)$ define the p-metric as the distance function d_p on $\mathbb{R}^n$ such that:

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}$$
$$d_\infty(x, y) = \sup_{i \in \{1, \ldots, n\}} \{|x_i - y_i|\}$$

[8] This section was directly copied from my 812A notes with maybe minor additions.

Definition (sequence space l_p): The **sequence space** l_p is the space containing infinite real sequences that satisfy:

$$(x_k) \in l_p \Rightarrow \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} < \infty$$
$$(x_k) \in l_\infty \Rightarrow \sup_{i \in \{1,\dots,n\}} \{|x_i|\} < \infty$$ [9]

These only discuss distances on sets with up to countably infinitely many points. For cases of distances with uncountably infinitely many points, we use the function spaces L^p. For example, let $\mathbf{C}[0,1]$ denote all continuous real functions on the interval [0,1]. $\forall\, p \in [1, \infty)$, $f, g \in \mathbf{C}[0,1]$, we have

$$d_p(f,g) \equiv \left(\int_0^1 |f(t) - g(t)|^p dt \right)^{\frac{1}{p}}$$
$$d_\infty(f,g) \equiv \underbrace{\max_{t \in [0,1]} |f(t) - g(t)|}_{\text{Since } f,g \text{ are continuous, this is equivalent to taking the supremum}}$$

Proposition: P-metric is valid $\forall p \in [1, \infty)$ on $\mathbb{R}^n$ and the l_p space.

Theorem 2.1: Minkowski's Inequality

$\forall p \in [1, \infty)$, $\forall n \in \mathbb{N}$, $(x_k)_{k=1}^{\infty}, (y_k)_{k=1}^{\infty} \in \mathbb{R}^\infty$, $i = 1, \dots, n$,

$$\left(\sum_{i=1}^{n} |x_k + y_k|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |x_k|^p \right)^{\frac{1}{p}}$$

Definition (Distance): Given a metric space $M \equiv (X, d_x)$ and $x \in X, A \subseteq X$, we define the **distance** between a point $x \in X$ and a set $A \subseteq X$ to be:

$$d_X(x, A) \equiv \inf_{a \in A} d_X(x, a)$$

Definition: Given a metric space $M \equiv (X, d_x)$, $\forall x_0 \in X, \forall\, \varepsilon \in \mathbb{R}_{++}$, the $\varepsilon-$neighborhood of x_0 is defined as:

$$N_\varepsilon^{d_X}(x_0) \equiv \{x \in X \mid d(x, x_0) < \varepsilon\}$$

Definition: Given a metric space $M \equiv (X, d_x)$, $\forall A \subset X, \forall \varepsilon \in \mathbb{R}_{++}$, the $\varepsilon-$neighborhood of A is defined as:

$$N_\varepsilon^A(x_0) \equiv \{x \in X \mid \exists a \in A,\, d(x, x_0) < \varepsilon\} = \bigcup_{a \in A} N_\varepsilon^{d_X}(a)$$

Definition (Open Set): Given a metric space $M \equiv (X, d_x)$, $\forall O \subseteq X$, we say that O is **open** in X with respect to d_X if $\forall x \in O$, $\exists \varepsilon \in \mathbb{R}_{++}$, $N_\varepsilon^{d_X}(x) \subseteq O$

(**Important**) **Definition (Closed Set):** Given a metric space $M \equiv (X, d_x)$, $S \subseteq X$ is **closed** if $X \setminus S$ is open.

Equivalent Definition for closed-ness: Given a metric space $M \equiv (X, d_x)$, $S \subseteq X$ is closed if every convergent sequence in S converges to a point in S.

Set opertions and open/closedness:

- Every *union* of *open sets* is open
- Every *intersection* of *closed sets* is closed
- Every finite *union* of *closed sets* is closed
- Every finite *intersection* of *open sets* is open

Proof: Set Operations and Open/Closed-ness

Every union of open sets is open.
Suppose otherwise that this is not true and that there exists some open sets $A, B \subset X$ such that $A \cup B$ is not open. Then $\exists a \in A \cup B$ such that $\forall \varepsilon > 0$, $\exists c \in A^c \cap B^c$ such that $c \in N_\varepsilon^d(a)$. Since $a \in A \cup B$, a is in at least one of A or B. and c is in neither A or B. But that means whichever set a is from, that set is not an open set, which is a contradiction.

Every intersection of closed sets is closed.
Given two closed set $A, B \subset X$, their complements must be open. i.e., A^c, B^c are open. Since A^c, B^c are open, their union is also open. i.e.,$A^c \cup B^c = (A \cap C)^c$ is open. Since $(A \cap C)^c$ is open, by definition, $A \cap C$ is closed.

Every finite intersection of open sets is open

Take n open sets $A_1, A_2, \dots, A_n \subseteq X$, we first show that $\bigcap_{i=1}^{n} A_i$ is open. i.e., the finite intersection of open sets is open.

Since all A_i, $i \in \{1, 2, \dots, n\}$ are open, we know that $\forall a_i \in A_i$, $\exists\, \varepsilon_i > 0\, s.t.\, N^d_{\varepsilon_i}(a_i) \subseteq A_i$. Then $\forall a \in \bigcap_{i=1}^{n} A_i$ we know $a \in A_i, \forall i \in \{1, 2, \dots, n\}$. Take $\varepsilon^* = \min_{i \in \{1,2,\dots,n\}} \varepsilon_i$, then we know that $N^d_{\varepsilon^*}(a) \subseteq \bigcap_{i=1}^{n} A_i$.

To show that the infinite case fails, take $A_i = (-\infty, \frac{1}{n}) \cup (1, \infty) \subset \mathbb{R}$, $n \in \mathbb{N}$, then $\bigcap_{i=1}^{\infty} A_i = (\bigcup_{i=1}^{\infty} A_i)^c \to \mathbb{R} \setminus (0, 1] = (-\infty, 0] \cup (1, \infty)$. At $0 \in \bigcap_{i=1}^{\infty} A_i$, take any $0 < \varepsilon < 1$, we can see that $0 + \varepsilon = \varepsilon \notin \bigcap_{i=1}^{\infty} A_i$, hence the $\bigcap_{i=1}^{\infty} A_i$ is not closed.

Every finite union of closed sets is closed

By definition of closed sets, their complement is open. Take n closed sets $A_1, A_2, \dots, A_n \subseteq X$, we know that $A_1^c, A_2^c, \dots, A_n^c \subseteq X$ are open, and we know, from the previous proof, that $\bigcap_{i=1}^{n} A_i^c$ are open as well. Let the finite union of these closed sets be $A = \bigcup_{i=1}^{n} A_i$, and its complement, by De Morgan's Law is $A^c = \bigcap_{i=1}^{n} A_i^c$. Since A^c is open, by definition, the set $A = \bigcup_{i=1}^{n} A_i$, the finite union of n closed sets, is closed.

To show that the infinite case fails, take $A_i = [\frac{1}{i}, 1]$, then $\bigcup_{i=1}^{\infty} A_i \to (0, 1]$. But the sequence $\frac{1}{n}$ converges to $0 \in (0, 1]$, making the union a non-closed set.

□

Definition: Given a metric space $M \equiv (X, d_x)$, $\forall Y \subseteq X$,

$x \in X$ is a **boundary point** of Y if $\forall \varepsilon \in \mathbb{R}_{++}$, $N^{d_X}_{\varepsilon}(x) \cap Y \neq \varnothing \wedge N^{d_x}_{\varepsilon}(x) \cap Y^c \neq \varnothing$

$x \in X$ is an **interior point** of Y if $\exists \varepsilon \in \mathbb{R}_{++}$, $N^{d_X}_{\varepsilon}(x) \subseteq Y$

$x \in X$ is a **closure point** of Y if $\forall \varepsilon > 0$, $\exists y \in Y\ s.t.\ y \in N^{d_X}_{\varepsilon}(x)$

Another Equivalent Definition for closed-ness: Given a metric space $M \equiv (X, d_x)$, $S \subseteq X$ is closed if it contains all of its closure points.

Definition (Interior): Given a metric space $M \equiv (X, d_x)$. The **interior** of $Y \subseteq X$ (denoted as $int(Y)$) is defined as (the union of all interior points):

$$int(Y) \equiv \bigcup \{O \subseteq Y \mid O \text{ is open}\}$$

Definition (Closure): Given a metric space $M \equiv (X, d_x)$. The **closure** of $Y \subset X$ (denoted as $cl(Y)$) is defined as (the smallest closed set that contains Y):

$$cl(Y) \equiv \bigcap \{S \subseteq X \mid Y \subseteq S, S \text{ is closed}\}$$

Alternatively, $cl(Y) = Y \cup \{\text{limit points of } Y\}$

Definition (Boundary): Given a metric space $M \equiv (X, d_x)$. The **boundary** of $Y \subseteq X$ (denoted as $bd(Y)$) is defined as:

$$bd(Y) = cl(Y) \setminus int(Y)$$

Sequential Definition of a Limit: Given a metric space $M \equiv (X, d_x)$, $(x_k)_{k=1}^{\infty}$ on X, (x_k) is said to converge to x ($\lim_{k \to \infty} x_k = x \in X$) if $\forall \varepsilon \in \mathbb{R}_{++}, \exists m \in \mathbb{N}\ s.t.\ \forall k \geq m,\ d(x_k, x) < \varepsilon$

Definition: Given a metric space $M \equiv (X, d_x)$ the lim sup and lim inf of a sequence $(x_k)_{k=1}^{\infty}$ are defined as:

$$\limsup (x_k)_{k=1}^{\infty} \equiv \lim_{n \to \infty} (\sup\{x_k \mid k \geq n\})$$
$$\liminf (x_k)_{k=1}^{\infty} \equiv \lim_{n \to \infty} (\inf\{x_k \mid k \geq n\})$$

Theorem 2.2: Sequence Convergence

Given a metric space $M \equiv (X, d_x)$. $(x_k)_{k=1}^{\infty}$ is convergent if and only if

$$\limsup(x_k) = \liminf(x_k) = \lim_{k \to \infty} (x_k)$$

Theorem 2.3: Limit Point is Unique

Given a metric space $M \equiv (X, d_x)$.

$$\lim_{k \to \infty} (x_k) = x \in X \wedge \lim_{k \to \infty} (x_k) = y \in X \iff x = y$$

Proof 2.3

Suppose otherwise that there exists a convergence sequence $\{x_n\} \in \mathbb{R}^n$ that has 2 limit points x and x'. Then $\forall \varepsilon > 0, \exists N, N' \in \mathbb{N}$ such that $\forall n > N, \|x_n - x\| < \varepsilon$ and $\forall n > N', \|x_n - x'\| < \varepsilon$.
Take $\varepsilon^* = \frac{1}{4}\|x - x'\|$ and $N^* = \max\{N, N'\}$. Then we have $\forall n \geq N^*$, $\|x_n - x\| < \varepsilon^* = \frac{1}{4}\|x - x'\| = \frac{1}{4}\|x - x_n + x_n - x'\| \leq \frac{1}{4}\|x_n - x\| + \frac{1}{4}\|x_n - x'\| < \frac{1}{2}\varepsilon^*$
Since $\varepsilon^* > 0$, $\varepsilon^* < \frac{1}{2}\varepsilon^*$ is a contradiction. Hence a convergent sequence in $\mathbb{R}^n$ can only have one limit point. □

Theorem 2.4: Alternative Definition of Closedness

Given a metric space $M \equiv (X, d_x)$, $Y \subseteq X$ is closed if and only if every sequence in Y that converges in X also converges in Y. i.e.,

$$Y \text{ is closed} \iff [(y_k)_{k=1}^{\infty} \in Y^{\infty}, ((y_k) \to x \in X) \;\Rightarrow\; (x \in Y)]$$

Proof 2.4

" $\Rightarrow$ "

Assume that Y is closed. Take the sequence $(x_k)_{k=1}^{\infty} \in Y^{\infty}$ with $(x_k) \to x \in X$. Suppose otherwise that $x \in X \setminus Y$, then $X \setminus Y$ is closed and $\exists \varepsilon \in \mathbb{R}_{++}$ $s.t.$ $N_{\varepsilon}^{dx}(x) \subseteq X \setminus Y$. Since $(x_k) \to x$, $\lim_{k\to\infty} d(x_k, x) = 0$, $\exists n \in \mathbb{N}$, $x_n \in N_{\varepsilon}^{dx}(x)$, so $x_n \notin Y$, but by definition, $x_n \in Y$, so by contradiction, $x \in Y$.

"$\Leftarrow$"

Assume that for every sequence $(x_k)_{k=1}^{\infty} \in Y$, we have $(x_k)_{k=1}^{\infty} \to x \in X \;\Rightarrow\; x \in Y$. Suppose otherwise that Y is not closed, then $X \setminus Y$ is not open. Take $x \in X \setminus Y$ such that $\forall \varepsilon \in \mathbb{R}_{++}, N_{\varepsilon}^{dx}(x) \cap Y \neq \varnothing$. Then $\forall m \in \mathbb{N}$, $\exists x_m \in N_{\varepsilon m}^{dx}(x) \cap Y$. So $(x_m)_{m=1}^{\infty} \in Y^{\infty}$, and by assumption, $(x_m)_{m=1}^{\infty} \to x \in Y$. But by construction, $x \in X \setminus Y \iff x \notin Y$. Hence, by contradiction, Y has to be closed.[a] □

[a]The gist of this proof is that "If Y is not clsoed, then no limit $x \in Y$. Since we found that $x \in Y$, then Y must be closed.

Important: Open/Closed-ness always depends on the underlying metric space (X, d_X). For example, $(0, 1)$ is open in $(\mathbb{R}, d_1)$ but closed in $((0, 1), d_1)$

Important: Open/Closed-ness is not binary. Sets can be neither open nor closed, sets can also be both open and closed.

2.2 Properties of Well-Behaved Metric Spaces

1. Connectedness
2. Separability
3. Completeness
4. Boundedness
5. Compactness

2.2.1 Connectedness

Definition: Given a metric space $M \equiv (X, d_x)$. M is said to be **connected** if a subspace **cannot** be obtained without cutting the space (e.g., $[0, 1]$ is connected, but $[0, 1] \cup [2, 3]$ is not connected

Definition: Given a metric space $M \equiv (X, d_x)$. M is **connected** if there does **not** exist 2 non-empty, disjoint, and open subsets A, B such that $A \cup B = X$

Definition: Given a **connected** metric space $M \equiv (X, d_x)$, a subset $Y \subseteq X$ is connected in X if Y is a connected metric subspace of X

2.2.2 Separability

Definition (Dense): Given a metric space $M \equiv (X, d_x)$. $Y \subseteq X$ is **dense** in X if $cl(Y) = X$

Definition (Separable): Given a metric space $M \equiv (X, d_x)$. X is **separable** if X contains a subset that is countable and dense.

Theorem 2.5: Weierstrass Approximation Theorem

$\forall\, a, b \in \mathbb{R}$, the set of all polynomial functions on $[a, b]$ is dense in $\mathbf{C}[0, 1]$

Corollary: $\mathbf{C}[a, b]$, the set of all continuous functions on $[a, b]$, is separable

Proof 2.5

The set of rational polynomials is countable since there are finitely many terms (for a polynomial) and the coefficients are rational. The closure will include irrational coefficient polynomials given completeness in $\mathbb{R}$, so $\mathbf{C}[0, 1]$ is dense in $\mathbb{R}$ □

Theorem 2.6

Given a metric space $M \equiv (X, d_x)$ is separable. There exists a countable class $\mathbb{O}$ of open sets in X such that $\forall$open $U \subseteq X, U = \bigcup\{O \in \mathbb{O} \mid O \subset U\}$

Proof 2.6

Since (X, d_X) is separable, take $Y \subseteq X$ be a countable and dense subset of X. Define $\mathbb{O} \equiv \{N_\varepsilon(z) \mid z \in Y, \varepsilon \in \mathbb{Q}_{++}\}$ (Notice that $\mathbb{O}$ is a countable set of open sets). Take an open subset $U \subseteq X$ and $x \in U$, we want to show that $x \in O$ for some $O \in \mathbb{O}$ such that $O \subseteq U$. Since U is open, $\exists \varepsilon \in \mathbb{Q}_{++}$ such that $N_\varepsilon(x) \subseteq U$. Then since $cl(Y) = X$ (because (X, d_x) is separable), $\exists y \in Y$ with $d(x, y) < \frac{\varepsilon}{2}$. i.e., $x \in N^{d_X}_{\frac{\varepsilon}{2}}(y) \subseteq N^{d_X}_{\varepsilon}(x) \subseteq U$. Since $y \in Y \wedge x \in N^{d_X}_{\frac{\varepsilon}{2}}(y)$, we know $x \in N^{d_X}_{\frac{\varepsilon}{2}}(y) = O \in \mathbb{O}$ □

2.2.3 Completeness

Definition: Given a metric space $M \equiv (X, d_x)$. The sequence $(x_k)_{k=1}^{\infty}$ is a **Cauchy Sequence** if $\forall \varepsilon \in \mathbb{R}_{++}$, $\exists m_\varepsilon \in \mathbb{N}$ such that $\forall\, j, k \in \{h \in \mathbb{N} \mid h > m_\varepsilon\}$, $d_X(x_j, x_k) < \varepsilon$

Definition (Completeness): Given a metric space $M \equiv (X, d_x)$. M is **complete** if every Cauchy sequence in X converges in X

2.2.4 Boundedness

Definition3 (Bounded): Given a metric space $M \equiv (X, d_x)$. The set $S \subseteq X$ is said to be **bounded** if $\exists \varepsilon \in \mathbb{R}_{++}, x \in X \ s.t. \ X \subseteq N_\varepsilon^{d_X}(x)$

2.2.5 Compactness (Super Important)

Definition: Given a metric space $M \equiv (X, d_x)$, $Y \subseteq X$. A collection $\mathbb{O}$ of (open) subsets of X is said to be a(n) "(open) cover" of Y if $Y \subseteq \bigcup_{o_i \in \mathbb{O}} o_i$

(Important) **Definition (Compactness):** Given a metric space $M \equiv (X, d_x)$. M is compact if **every open cover** of X has a finite subset that is also an open cover of X

Definition: Given a metric space $M \equiv (X, d_x)$, $Y \subseteq X$ is compact in X if every open cover of Y has a finite open sub-cover[10]

Definition: Given a metric space $M \equiv (X, d_x)$, $Y \subset X$ is **sequentially compact** if every sequence in Y has a subsequence that converges to a point in Y

Theorem 2.7

$Y \subseteq X$ is compact if and only if Y is sequentially compact in X

Theorem 2.8

$Y \subseteq X$ is compact $\Rightarrow$ Y is closed and bounded.

Proof 2.8

Assuming that Y is compact, we need to show that $X \setminus Y$ is open (so Y is closed). If $X \setminus Y = \varnothing$, then closed and bounded is trivially true or false since we don't know anything about X. So let's assume that $Y \subset X$ so that $X \setminus Y \neq \varnothing$
Take $x \in X \setminus Y$, then $\forall y \in Y$, $\exists\, \varepsilon_y = \frac{d_X(x,y)}{2} \in \mathbb{R}_{++}$ such that $N_{\varepsilon_y}^{d_X}(x) \cap N_{\varepsilon_y}^{d_X}(y) = \varnothing$. Moreover, since $\{N_{\varepsilon_y}^{d_X}(y) \mid y \in Y\}$ is an open cover of Y and Y is compact, we know that there is a finite open subcover Z of $\{N_{\varepsilon_y}^{d_X}(y) \mid y \in Y\}$ such that $\{N_{\varepsilon_y}^{d_X}(y) \mid y \in Z, |Z| < \infty\}$ also covers Y. Now define $\varepsilon^* = \min_{y \in Z} \varepsilon_y$, then $N_\varepsilon^{d_X}(x) \in X \setminus Y$, hence $X \setminus Y$

[10] Note that this is not equivalent to having a finite open cover. Considering that the universe X is a finite open cover.

is open and so Y is closed. Since Y is compact and not equal to X, Y must also be bounded because $Y \subseteq N_{\tilde{\varepsilon}}^{d_X}(y), \varepsilon = \sum \varepsilon^*, y \in Y$ □

Theorem 2.9: A closed subset of a compact space is compact

Let $M \equiv (X, d_x)$ be a compact metric space. If $Y \subseteq X$ is closed, Y is compact.

Proof 2.9

Take $\mathbb{O}$ to be an open cover of Y, then $\mathbb{O} \cup \{X \setminus Y\}$ is an open cover of X [a]. Since X is compact, we know that $\mathbb{O} \cup \{X \setminus Y\}$ has a finite open subcover $\mathbb{O}'$ of X. Since $\mathbb{O}'$ is finite, $\mathbb{O}' \setminus \{X \setminus Y\}$ is also finite. By construciton, $\mathbb{O}' \setminus \{X \setminus Y\}$ is an open subcover of $\mathbb{O}$ of Y. So $\mathbb{O} \cup \{X \setminus Y\}$ is a finite open subcover of Y and hence Y is compact. □

[a] Since Y is closed, $X \setminus Y$ is open.

Theorem 2.10: Heine-Borel Theorem

In the standard $(\mathbb{R}^n, d_2)$ metric space, take any set $Y \subset \mathbb{R}^n$, then

$$Y \text{ is compact} \iff Y \text{ is closed and bounded}$$

Proof 2.10

First we need to show that if $Y = \mathbb{R}^n$, Y cannot be compact in $(\mathbb{R}^n, d_2)$. Take the sequence of open sets $(-k, k)^n \subset \mathbb{R}^n$, then $\bigcup_{k=1}^{\infty} (-k, k)^n$ is open, and more importantly, an open cover of $\mathbb{R}^n$. Notice that $\bigcup_{k=1}^{\infty} (-k, k)^n$ does not have any finite collection of subsets that also cover $\mathbb{R}^n$ as $\bigcup_{k=1}^{\alpha} (-k, k)^n$, $\alpha < \infty$ does not cover $\mathbb{R}^n$. Hence we only need to prove the statement for $Y \subset \mathbb{R}^n$

"$\Rightarrow$":

Assume that $S \subset \mathbb{R}^n$ is compact in $(\mathbb{R}^n, d_2)$. We want to show that $\mathbb{R}^n \setminus S$ is an open set and that S is bounded. Take $x \in \mathbb{R}^n \setminus S$ and $s \in S$, and take $\varepsilon_s = \frac{d(x,y)}{2}$, then we know $N_{\varepsilon_s}^{d_2}(x) \cap N_{\varepsilon_S}^{d_2}(s) = \varnothing$. We also know that $\{N_{\varepsilon_s}^{d_2}(s) \mid s \in S\}$ is an open cover for S. Since S is compact, $\exists \mathbb{O} \subset \{N_{\varepsilon_s}^{d_2}(s) \mid s \in S\}$, $|\mathbb{O}| < \infty$, $S \subseteq \bigcup \mathbb{O}$. Take $\varepsilon^* = \min_{o \in \mathbb{O}} \varepsilon_o$,

then $N^{d_2}_{\varepsilon_o}(x) \subset N^{d_2}_{\varepsilon_s}(x) \subseteq \mathbb{R}^n \setminus S$. Since $x \in \mathbb{R}^n \setminus S$ is taken arbitrarily, we know $\mathbb{R}^n \setminus S$ is open and thus S is **closed**. Moreover, take $\varepsilon' = \sum_{o \in \mathbb{O}}$, since $\mathbb{O}$ is finite, we know $\varepsilon' < \infty$ and that $\forall\, s \in S$, $S \subseteq N^{d_2}_{e'}(s)$, so S is also **bounded**.
"$\Leftarrow$":
Assume that S is closed and bounded. Now supposed otherwise that S is not compact, then it is not sequentially compact. i.e., $\exists (s_n) \in S$ such that all subsequences $(s_{n_i}) \in S$ either does not converge or converge outside of S. But that would mean there is a sequence in S that converges outside of S, so S is not a closed set. Moreover, suppose that S is closed but not compact. Since S is not compact, there exists an open cover $\mathbb{O}'$ for S that does not have a finite set of open sub-covers. So $\exists s' \in S, \forall k < \infty, s' \notin \bigcup_{i=1}^{k} \{o_i \in \mathbb{O}'\}$. In other words, $\forall \varepsilon > 0, S \nsubseteq N^{d_2}_{\varepsilon}(s')$, meaning that S is not bounded. $\square$

2.2.6 Continuity of Functions

Definition: Given two metric spaces (X, d_x), (Y, d_Y), and the function $f : X \to Y$. We say that f is **continuous** on X at $x_0 \in X$ if $\forall \varepsilon \in \mathbb{R}_{++}$, $\exists \delta_\varepsilon(x_0) \in \mathbb{R}_{++}$ such that

$$\forall x \in X, d_X(x, x_0) < d_\varepsilon(x_0) \;\Rightarrow\; d_Y(f(x), f(x_0)) < \varepsilon$$

or equivalently,

$$\forall x \in N^{d_X}_{\delta_\varepsilon}(x_0),\; f(x) \in N^{d_Y}_{\varepsilon}(f(x_0))$$

Equivalent Definition: Given a metric space $(X, d_x), (Y, d_Y)$. A function $f : X \to Y$ is continuous at $x_0 \in X$ if and only if $\forall$ open $O \subseteq Y\, s.t.\, f(x_0) \in O$, $\exists \delta \in \mathbb{R}_{++}\, s.t. \forall x \in N^{d_X}_{\delta}(x_0),\; f(x) \in O$

Definition: Given a metric space $M \equiv (X, d_x)$. $f : X \to \mathbb{R}$ is **upper-semi-continuous** at $x_0 \in X$ if $\forall \varepsilon \in \mathbb{R}_{++}, \exists \delta_\varepsilon \in \mathbb{R}_{++}$ such that

$$\forall x \in X,\; d_X(x, x_0) < \delta_\varepsilon \;\Rightarrow\; f(x) \leq f(x_0) + \varepsilon$$

or equivalently

$$\limsup_{x \to x_0} f(x) \leq f(x_0)$$

Equivalent Definition: Given a metric space $M \equiv (X, d_x)$. $f : X \to \mathbb{R}$ is **upper-semi-**

continuous at $x_0 \in X$ if the set $\{x \in X \mid f(x) \geq x_0\}$ is closed.

Definition: Given two metric space $(X, d_X), (Y, d_Y)$. $f : X \to \mathbb{R}$ is **uniformly continuous** if $\forall \varepsilon \in \mathbb{R}_{++}, \exists \delta_\varepsilon \in \mathbb{R}_{++}$ such that:

$$\forall x, x_0 \in X, d_X(x, x_0) < \delta_\varepsilon \;\Rightarrow\; d_Y(f(x), f(x_0)) < \varepsilon$$

2.2.7 Continuity of Correspondences

Recall the open set definition of functional continuity. We want to keep using a similar definition but expand to correspondences. What can we do?

Definition (Upper Hemi-Continuity): Given metric spaces $(X, d_x), (Y, d_Y)$. The correspondence $g : X \rightrightarrows Y$ is **upper-hemi-continuous (*uhc*)** at $x_0 \in X$ if and only if $\forall$ open $O \subseteq Y$ $s.t.$ $g(x) \subseteq O, \exists\, \delta_O \in \mathbb{R}_{++}\, s.t.\, g(N^{d_X}_{\delta_O}(x_0)) \subseteq O$

Definition: Given metric spaces $(X, d_x), (Y, d_Y)$. The correspondence $g : X \rightrightarrows Y$ is **closed-valued** at $x_0 \in X$ if $\forall x \in X,\ g(x) \subseteq Y$ is a closed set.

Definition: Given metric spaces $(X, d_x), (Y, d_Y)$. The correspondence $g : X \rightrightarrows Y$ is **compact-valued** at $x_0 \in X$ if $\forall x \in X,\ g(x) \subseteq Y$ is a compact set.

Proposition: Given metric spaces $(X, d_x), (Y, d_Y)$, the *compact-valued* correspondence $g : X \rightrightarrows Y$, and $x_0 \in X$. If for every sequence $(x_k)_{k=1}^\infty \to x_0$ and every sequence $(y_k)_{k=1}^\infty$ such that $y_k \in g(x_k)$, there exists a convergent subsequence $(y_{k_j})_{j=1}^\infty \to y \in g(x_0)$, then g is **upper-hemi-continuous** at $x_0 \in X$ (See Efe Ok p.288 for pictoral representation)

Definition: Given metric spaces $(X, d_x), (Y, d_Y)$ and the correspondences $g : X \rightrightarrows Y$. The **graph** of g is the set $gr(g) \equiv \{(x, y) \in X \times Y \mid y \in g(x)\} \subseteq X \times Y$

Definition: Given metric spaces $(X, d_x), (Y, d_Y)$ and the correspondences $g : X \rightrightarrows Y$. We say that g has a **closed graph** in $X \times Y$ if $gr(g)$ is closed in $(X \times Y, d_{X\times y})$
Remark: Given metric spaces $(X, d_x), (Y, d_Y)$. The standard metric in Cartesian product is called the **product-metric** and is defined as

$$d_{X\times Y} : (X \times Y) \times (X \times Y) \to \mathbb{R}_+$$
$$d_{X\times Y}\left((x, y), (x', y')\right) = d_X(x, x') + d_Y(y, y')$$

This is analogous to using d_1 on $\mathbb{R}^2$.

Definition: Given metric spaces $(X, d_x), (Y, d_Y)$ and the correspondences $g : X \rightrightarrows Y$. We say that $g : X \rightrightarrows Y$ is **closed** at $x_0 \in X$ if $\forall\, ((x_k, y_k))_{k=1}^{\infty}, x_k \in X, y_k \in g(x_k), \forall k \in \mathbb{N}, ((x_k, y_k))_{k=1}^{\infty} \to (x_0, y_0)$, we have $y_0 \in g(x_0)$. In other words, a closed graph is a graph of g that is closed at every point in the graph.[11]

Proposition: Given metric spaces $(X, d_x), (Y, d_Y)$ and the correspondences $g : X \rightrightarrows Y$. If $g : X \rightrightarrows Y$ is **uhc AND closed-valued**, then it has a closed graph.

Proposition: Given metric spaces $(X, d_x), (Y, d_Y)$ and the correspondences $g : X \rightrightarrows Y$. If Y is compact and g has a closed graph, then g is **upper-hemi-continuous** everywhere on X.

Definition (Lower Hemi-Continuity): Given metric spaces $(X, d_x), (Y, d_Y)$. The correspondences $g : X \rightrightarrows Y$ is **lower-hemi-continuous** at $x_0 \in X$ if

$$\forall y \in g(x_0), \forall (x_k)_{k=1}^{\infty} \to x_0 \text{ with } x_k \in X, \forall k \in \mathbb{N},\ \exists (y_k)_{k=1}^{\infty} \to y_0 \ s.t.\ y_k \in g(x_k),\ \forall k \in \mathbb{N}$$

In other words, g is lower-hemi-continuous at $x_0 \in X$ if for all $y \in g(x_0)$, every sequence in X that converges to x_0 has a corresponding sequence in Y that converges to y. Notice that this is very different from the sequential characterization of *uhc* because it requires **every point** in the image to have convergent sequences from **all directions** of the domain.

Equivalent Definition (Open sets): Given metric spaces $(X, d_x), (Y, d_Y)$. The correspondence $g : X \rightrightarrows Y$ is **lower-hemi-continuous** at $x_0 \in X$ if

$$\forall \text{ open } O \subseteq Y \text{ such that } g(x_0) \cap O \neq \varnothing \ \Rightarrow\ \forall x \in N_{\varepsilon}^{d_X}(x_0),\ g(x) \cap O \neq \varnothing$$

Definition: Given metric spaces $(X, d_x), (Y, d_Y)$. The correspondences $g : X \rightrightarrows Y$ is **continuous** at $x_0 \in X$ if and only if it is **both** uhc and lhc at x_0.
Proposition: Given metric spaces $(X, d_x), (Y, d_Y)$ and a continuous function $f : X \to Y$. If $Z \subseteq X$ is a compact set, then $f(Z) \subseteq Y$ is also a compact set.

[11] Being closed and being closed-valued are not equivalent.

Theorem 2.11: Weierstrass Maximum Theorem

Given a metric space $M \equiv (X, d_x)$ and $A \subseteq X$ a compact subset of X. If $f : A \to \mathbb{R}$ is a continuous function, then $\exists a_{max}, a_{min} \in A$ such that $f(a_{max}) = \sup\{f(a)\}$ and $f(a_{min}) = \inf\{f(a)\}$.
Notationally, a_{min} and a_{max} are defined as:

$$a_{min} \equiv \underset{a \in A}{\text{argmin}} f(a)$$
$$a_{max} \equiv \underset{a \in A}{\text{argmax}} f(a)$$

In other words, a function is continuous over a compact set, there exists maximizers and minimizers of the function that obtains extreme values within the compact range of the function.

This theorem allows to know that if we have a compact domain (think budget set) and a continuous utility function, then we must have a utility maximizing bundle.

Proof 2.11: Weierstrass Maximum Theorem

To show that $\exists a_{max} \in A$ such that $f(a_{max}) = \sup\{f(a)\}$, construct a sequence $(a_k)_{k=1}^{\infty}$ such that $\forall i, j \in \mathbb{N}\ s.t.\ a_i \neq a_j, i < j, \iff f(a_i) \leq f(a_j)$ (i.e., construct a sequence so that the sequence $f(x_k)$ is strictly increasing). Then since A is assumed to be compact, there is a convergent subsequence $(a_{ki})_{i=1}^{\infty}$ of $(a_k)_{k=1}^{\infty}$ that converges to a point $a \in A$. Since f is continuous, we know that $f(a_{ki}) \to f(a) \in f(A) = \{x \in \mathbb{R} \mid x = f(a), a \in A\}$. By the construction of the sequence $(a_k)_{k=1}^{\infty}$, we know that $\sup(\{f(a)\} = f(a) \in f(A)$ and $a = a_{max}$

Similarly, to show that $\exists a_{min} \in A$ such that $f(a_{min}) = \inf(\{f(a)\}$, we construct a sequence $(a'_k)_{k=1}^{\infty}$ such that $\forall i, j \in \mathbb{N}\ s.t.\ a_i \neq a_j, i < j \iff f(a'_i) \geq f(a'_j)$. Then since A is assumed to be compact, there is a convergent subsequence $(a'_{ki})_{i=1}^{\infty}$ of $(a'_k)_{k=1}^{\infty} \to a' \in A$. Since f is continuous, we know that $f(a'_{ki}) \to f(a') \in f(A) = \{x \in \mathbb{R} \mid x = f(a), a \in A\}$. By the construction of the sequence $(a'_k)_{k=1}^{\infty}$, we know that $sup(\{f(a)\} = f(a') \in f(A)$ and $a' = a_{min}$ □

Theorem 2.12: Berge's Theorem of Maximum

Let $X, Y \subseteq \mathbb{R}^n$ be non-empty sets. Let Y be a compact set and $f : X \times Y \to R$ be a continuous function. Let $V(x) \equiv \max_{y \in Y} f(x, y)$ and $y^*(x) = \operatorname{argmax}_{y \in Y} f(x, y)$, then $V : X \to \mathbb{R}$ is continuous and $y^*(x)$ is upper-hemi-continuous.

2.3 Fixed Point Theorems

Definition (Fixed Point): Let X be a non-empty set and $g : X \rightrightarrows X$ a self-maping correspondence. We say that a point $x \in X$ is a fixed point of g if $x \in g(x)$. If g is a self-mapping function instead, $x \in X$ is a **fixed point** if $g(x) = x$.

Theorem 2.13: Tarski's Fixed Point Theorem

$\forall n \in \mathbb{N}, \forall f : [0,1]^n \to [0,1]^n$

$$f_k : [0,1]^n \to [0,1] \text{ is non-decreasing } \Rightarrow f \text{ attains a fixed point}$$

Theorem 2.14: Brouwer's Fixed Point Theorem

Let $x \subseteq \mathbb{R}^n$ be a non-empty, compact, and convex subset of $\mathbb{R}^n$. Any continuous self-mapping $f : X \to X$ attains a fixed point in X.

Proof 2.14: Baby Brouwer

We will prove Brouwer for the case of $[0,1] \subset \mathbb{R}^1$.
Let $g(x) = x - f(x)$, we want to show that $\exists x \in [0,1]$ such that $g(x) = 0$. WLOG, suppose that $g(0) = 0 - f(0) < 0$ and $g(1) = 1 - f(1) > 0$. Since x and $f(x)$ is continuous, by the *Intermediate Value Theorem*, $\exists c \in [0,1]$ such that $g(0) < g(c) = 0 < g(1)$. This means that $f(c) = c$ and hence c is a fixed point of f. □

Definition: Take $X \subseteq \mathbb{R}^n$, $f : X \to X$ is a **contraction** if $\exists \lambda \in (0,1) \subseteq \mathbb{R}_{++}$ such that

$$\forall x, y \in X, \ \|f(x) - f(y)\| \leq \lambda \|x - y\|$$

Theorem 2.15: Banach's Fixed Point Theorem (Contraction Mapping)

Take $X \subseteq \mathbb{R}^n$ to be a closed subset of $\mathbb{R}^n$. If $f : X \to X$ is a contraction, then f has a *unique* fixed point in X.

Theorem 2.16: Kakutani's Fixed Point Theorem

Take $X \subseteq \mathbb{R}^n$ to be compact and convex. Let $g : X \rightrightarrows X$ to be convex-valued, closed-valued, and upper-hemi-continuous. Then the correspondence g has a fixed point in X.

Remark: Notice that the common theme here is a self-mapping on a compact and convex set. Based on the assumptions required, we have different strength of results. When using these for economic applications, make sure to make note of the assumptions available to you and use the results accordingly. Also note that these theorems, with the exception of Banach, only provides existence but not uniqueness. Later on (in Macro II), you will learn about the *Blackwell Conditions* for *Contraction Mapping Theorems* for more complicated cases.

Theorem 2.17: Contraction Mapping Theorem

Let (X, d_X) be a *complete metric space* and $f : X \to X$ is a contraction, then

(i) $\exists x^* \in X$ such that $f(x^*) = x^*$

(ii) $(x_k)_{k=1}^{\infty} \in X$, $x_{k+1} = f(x_k) \;\Rightarrow\; (x_k)_{k=1}^{\infty} \to x^*$

3 Dynamic Programming

3.1 Finite Horizon Economy ($t \in \{0, \dots, T\}$)

Recall from last semester that our representative consumers maximize

$$E\left[\sum_{t=0}^{T} \beta^t u(\underbrace{x_t}_{\text{State Var.}}, \underbrace{a_t}_{\text{Control Var.}})\middle| t=0\right]$$

Where the state variables follow the **transition function**

$$x_{t+1} = f(x_t, a_t, \varepsilon_t)$$

where ε_t is a stochastic process with a conditional distribution $F(\varepsilon_t | x_t, a_t)$.

In each period, the set of *control variables* is the result of a *decision* made based on the *state variables*. Formally, we can write the **optimal decision rules** as a vector of functions mapping from the state variable space to the control variable space: $\forall t \in \{1, \dots, T\}$,

$$a_t = \alpha(x_t),\ \pi_t \equiv (\alpha_0, \alpha_1, \dots, \alpha_T) \equiv \text{ Optimal Policy of Length } T$$

We use Pi_T to denote all feasible policy of length T. Notice that, so long as we keep our basic assumptions consistent (no incomplete markets, etc.), the decision rule should not change from period to period, so we can take away the time subscripts on the α functions.

$$\Pi_T = \{\pi_T = (\alpha(x_0), \alpha(x_1), \dots, \alpha(x_t) \mid \alpha \in \Gamma(x_t)\}$$

With this, we can define our value function ending at period-T. The value function is the **lifetime expected value** from the representative consumer's perspective at period 0.

$$W_T(x_), \pi_T) = E\left[\sum_{t=0}^{\infty} \beta^t u(x_t, \alpha(x_t) \mid t=0\right]$$

The representative agent hence solves the lifetime utility maximization problem by choosing a policy path that maximizes their value function. Hence we shall define the **Optimal Value Function**:

$$V_T(x_0) = \max_{\pi_T \in \Pi_T} W_T(x_0, \pi_T) = \max_{\pi_T \in \Pi_T} E\left[u(x_0, a_0) + E\left[\sum_{t=1}^{T} \beta^t u(x_t, a_t)\middle| t=1\right]\middle| t=0\right]$$

$$= \max_{\pi_T \in \Pi_T} E \left[u(x_0, a_0) + \beta \cdot \underbrace{E\left[\sum_{t=0}^{T-1} \beta^t u(x_t, a_t) \middle| t = 1 \right]}_{V_{T-1}} \middle| t = 0 \right]$$
$$= \max_{a_0 \in \Gamma(x_0)} E[u(x_0, a_0) \mid t = 0] + \beta V_{T-1}(f(x_0, a_0))$$

This is how we get the **Bellman Equation** that we are, hopefully, familiar with:

$$V_T(x) = \max_{a \in \Gamma(x)} \{u(x, a) + \beta V_{T-1}(f(x, a, \varepsilon))\}$$

Now that the Bellman equation is defined, we can continue with dynamic programming:

Step 1: $T = 0$ (Optimization with 0 period left to go)

$$\max_{a \in \Gamma(x)} u(x, a) \;\Rightarrow\; a(x_T) \;\Rightarrow\; V_0(x_T) = u(x_T, a(x_T)) \;\Rightarrow\; \alpha_T$$

Step 2: $T = 1$ (Optimization with 1 period left to go)

$$V_1(x) = \max_{a \in \Gamma(x)} \{u(x, a) + \beta V_0(f(x, a, \varepsilon))\}$$
$$= \max_{a \in \Gamma(x)} \{u(x, a) + \beta E[u(x_T, a(x_T)]\}$$
$$\Rightarrow \alpha_{T-1}$$

We continue this process iteratively and solve for the *optimal policy path* π_T.

3.2 Infinite Horizon Economy (with Discrete Time)

We have the value function:

$$V(x) = \max_{a \in \Gamma(x)} \{u(x, a) + \beta E[V(f(x, a, \varepsilon))]\}$$

and we, technically, have to now find the optimal policy path "infinitely". Lucky for us, with a little bit mathematics, we can know whether such policy path exists, and if it exists, what it looks like.

Known Result: If $\Gamma(x)$ is *continuous and compact*, the $u(x,a)$ is *continuous and bounded*, and the transition function $f(x,a,\varepsilon)$ is bounded, then a solution $(V(x))$ of the Bellman Equation exists and is continuous and bounded.

At "steady-state", our recursive problem can be simplified to:

$$V(a) = \max_{a\in\Gamma(x)}\{u(x,a) + \beta E_\varepsilon[V(f(x,a,\varepsilon)]\}$$

This looks eerily similar to a contraction, right? But we don't quite have that since we have no actual way to qualify $V(x)$. Here's an attempt:

Suppose that we want our *value function* to map from a compact subset $[a,b]$ of $\mathbb{R}$. Consider a function $\varphi \in \mathbf{C}[a,b]$ and a self-mapping $T : \mathbf{C}[a,b] \to \mathbf{C}[a,b]$ such that:

$$T(\varphi(x)) = \max_{a\in\Gamma(x)}\{u(x,a) + \beta E_\varepsilon[\varphi(f(x,a,\varepsilon)]\}$$

If T is a contraction on $\Gamma(x) \subseteq (\mathbf{C}[a,b], \sup\{|f-g|\})$, then the fixed point of T would thus become our optimized value function!

Proof: $T : \Gamma(x) \to \Gamma(x)$ is a contraction

Take two value functions $\varphi, \psi \in \Gamma(x) \subseteq \mathbf{C}[a,b]$, let $\bar{a} \equiv \underset{a}{argmax}\ \varphi(a(x))$. We can rewrite our value function as:

$$\begin{aligned}
T\circ\varphi &= u(x,\bar{a}) + \beta E_\varepsilon[\varphi(f(x,\bar{a},\varepsilon))] + \underbrace{\beta E_\varepsilon[\psi(f(x,\bar{a},\varepsilon))] - \beta E_\varepsilon[\psi(f(x,\bar{a},\varepsilon))]}_{=0} \\
&= \underbrace{\max_{a\in\Gamma(x)}\{u(x,a) + \beta E_\varepsilon[\psi(f(x,\bar{a},\varepsilon))]\}}_{T\circ\psi} + \beta E_\varepsilon[\varphi(f(x,\bar{a},\varepsilon))] - \beta E_\varepsilon[\psi(f(x,\bar{a},\varepsilon))] \\
&\le T\circ\psi + \sup\{\beta E_\varepsilon[|\varphi(f(x,\bar{a},\varepsilon)) - \psi(f(x,\bar{a},\varepsilon))|]\}
\end{aligned}$$

And by symmetry

$$\begin{aligned}
T\circ\psi &= u(x,\bar{a}) + \beta E_\varepsilon[\psi(f(x,\bar{a},\varepsilon))] + \underbrace{\beta E_\varepsilon[\varphi(f(x,\bar{a},\varepsilon))] - \beta E_\varepsilon[\varphi(f(x,\bar{a},\varepsilon))]}_{=0} \\
&= \underbrace{\max_{a\in\Gamma(x)}\{u(x,a) + \beta E_\varepsilon[\varphi(f(x,\bar{a},\varepsilon))]\}}_{T\circ\varphi} + \beta E_\varepsilon[\psi(f(x,\bar{a},\varepsilon))] - \beta E_\varepsilon[\varphi(f(x,\bar{a},\varepsilon))]
\end{aligned}$$

$$\leq T \circ \varphi + \sup\{\beta E_\varepsilon[|\psi(f(x, \bar{a}, \varepsilon)) - \varphi(f(x, \bar{a}, \varepsilon))|]\}$$

Combining the two inequalities, we get

$$\begin{aligned}
T \circ \varphi - T \circ \psi &\leq \sup\{\beta E_\varepsilon[|\varphi(f(x, \bar{a}, \varepsilon)) - \psi(f(x, \bar{a}, \varepsilon))|]\} \\
\Rightarrow \ |T \circ \varphi - T \circ \psi| &\leq \sup\{\beta E_\varepsilon[|\varphi(f(x, \bar{a}, \varepsilon)) - \psi(f(x, \bar{a}, \varepsilon))|]\} \\
\Rightarrow \ |T \circ \varphi - T \circ \psi| &\leq \beta \sup\{E_\varepsilon[|\varphi(f(x, \bar{a}, \varepsilon)) - \psi(f(x, \bar{a}, \varepsilon))|]\} \\
\Rightarrow \ |T \circ \varphi - T \circ \psi| &\leq \beta \sup\{|\varphi(f(x, \bar{a}, \varepsilon)) - \psi(f(x, \bar{a}, \varepsilon))|\} \\
\equiv d_\infty(T \circ \varphi, T \circ \psi) &\leq \beta d_\infty(\varphi, \psi)
\end{aligned}$$

So $T : \Gamma(x) \to \Gamma(x)$ is indeed a contraction mapping.

□

Since T is a contraction mapping on $\Gamma(x)$, we can numerically approach the optimal decision rule via:

Step 1:	We know that $\exists! V(x) \in Gamma(x)$ such that $T \circ V = V$
Step 2:	$\forall V_1(x)$, $V_2(x) = T \circ V_1(x) \ \Rightarrow$ get $\alpha_2(x)$
Step 3:	$V_3(x) = T \circ V_2(x) \ \Rightarrow$ get $\alpha_3(x)$
	$\vdots$
End Results	$\lim_{n\to\infty} V_n(x) = V(x)$, $\lim_{n\to\infty} \alpha_n(x) = \alpha(x)$

where $\alpha(x)$ is the optimal decision path.

Example:

Consider the set of continuous, increasing, and bounded functions C^I. We want to show that there is a "fixed point" $f^* \in C^I$ such that $T \circ f^* = f^*$

Take $\varphi \in C^I$, suppose $x_2 > x_1$, then $\varphi(x_2) > \varphi(x_1)$. We have

$$\begin{aligned}
T \circ \varphi(x_2) = \max_{a \in \Gamma(x_1)} &\{u(x_2, a) + \beta E[\varphi(f(x_2, a, \varepsilon))]\} \geq \\
&u(x_2, a) + \beta E[\varphi(f(x_2, a, \varepsilon)] \\
&u(x_1, \bar{a}) + \beta E[\varphi(f(x_1, \bar{a}, \varepsilon)] = T \circ \varphi(x_1)
\end{aligned}$$

xSee *Stokey & Lucas, Recursive Methods in Economic Dynamics* for more details.

Theorem 3.1: Blackwell's Theorem

Let $X \subset \mathbb{R}^n$ and $C(X)$ be the space of bounded functions $f : X \to \mathbb{R}$ with the sup-metric. Let $\varphi : C(X) \to C(X)$ be a self-mapping on this space. Then if,

(i) (**Monotonicity**) $\forall x \in X,\ f(x) \leq g(x) \ \Rightarrow\ \forall x \in X,\ B(f(x)) \leq B(g(x))$

(ii) (**Discounting**) $\exists \beta \in (0,1), \forall f \in C(X)$ and $a \geq 0$ such that

$$B(f(x)+a) \leq B(f(x)) + \beta a$$

Then B is a contraction with modulus β

Theorem 3.2: Leibniz's rule

Let f be a continuous function with a continuous partial derivative with respect to a parameter a. Let p, q be differentiable functions such that:

$$F(a) = \int_{p(a)}^{q(a)} f(x,a)dx$$

Then we have

$$\frac{\partial}{\partial a}F(a) = \int_{p(a)}^{q(a)} \frac{\partial}{\partial a} f(x,a)\, dx + f(q(a),a)\frac{\partial}{\partial a}q(a) - f(p(a),a)\frac{\partial}{\partial a}p(a)$$

Example: Uses 4.1.1 material

Given that the worker's reservation wage R is described by:

$$R = b + \frac{\beta}{1-\beta}\int_{R}^{\widehat{w}} w - R\, dF(w)$$

Show that the mapping T defined as:

$$T(R) = (1-\beta)b + \beta E[\max\{w, R\}]$$

is a contraction mapping.

Solution:
First, we want to show that $T(R)$ is increasing.

$$\begin{aligned} T(R) &= (1-\beta)b + \beta[\int_0^R R\,dF(w) + \int_R^{\widehat{w}} w\,dF(w)] \\ &= (1-\beta)b + \beta[R \cdot F(R) + \int_R^{\widehat{w}} w\,dF(w)] \end{aligned}$$

Differentiate $T(R)$ with respect to R using Leibniz's Rule:

$$\begin{aligned} \frac{d}{dR}T(R) &= \beta\left[F(R) + Rf(R)\right] + \beta \underbrace{\left[\int_R^{\widehat{w}} 0\,dF(w) + \widehat{w}f(\widehat{w}) \cdot 0 - f(R)R \cdot 1\right]}_{=\frac{\partial}{\partial R}\int_R^{\widehat{w}} w\,dF(w) = \frac{\partial}{\partial R}\int_R^{\widehat{w}} wf(w)\,dw \text{ (Leibniz's Rule)}} \\ &= \beta\left[Rf(R) + F(R) - Rf(R)\right] = \beta F(R) > 0 \end{aligned}$$

So we know that T is monotonically increasing, next, we want to show discounting. Take some $a > 0$:

$$\begin{aligned} T(R+a) &= (1-\beta)b + \beta \int_R^{\widehat{w}} w - R + a\,dF(w) \\ &= \underbrace{(1-\beta)b + \beta \int_R^{\widehat{w}} w - R\,dF(w)}_{T(R)} + \underbrace{\beta[1-F(R)]}_{<0} a \end{aligned}$$

Since $T(R)$ satisfy both monotonicity and discounting, by Blackwell's theorem, $T(R)$ as stated in the exercise is a contraction mapping.

4 Search Models

4.1 Simple Micro-Founded Models

4.1.1 Basic Job Search Model

In the infinite horizon economy, we will assume:

- The representative agent's problem is:

$$\max E\left[\sum_{t=0}^{\infty} \beta^t u(y_t)\right]$$

- For this model, we will assume that the agent's utility function is strictly increasing and concave so we can maximize wage instead
- At the beginning of each period, each unemployed agent gets a job offer with $w_t \sim F(w)$, $Supp(w_t) = [0, \widehat{w}]$, and the agent maximizes utility by deciding whether to take the offer or not
- If the agent is unemployed, they receive unemployment benefit b
- The transition function is

$$y_t = \begin{cases} w_t & \text{, if } a_t = 1 \text{ (accept the offer)} \\ b & \text{, if } a_t = 0 \text{ (reject the offer)} \end{cases}$$

  Note that this means our function space is non-convex ($a_t \in \Gamma(y_t) = \{0, 1\}$)

- After the agent accepted the job at $w_t = w$, they will receive w for each period infinitely (no quitting/layoffs)

Using BOPD, we can write down the agent's value functions:

- (Value of Decision about whether to work) $J(w_t) = \max\{\underbrace{V(w_t)}_{\text{Value of Working}}, \underbrace{U}_{\text{Value of Unemployment}}\}$
- (Value of Working) $V(w_t) = w + \beta V(w_t)$
- (Value of Unemployment) $U = b + \beta E[J(w_{t+1})]$

We can thus solve for the reservation wage in this economy. The reservation wage is the wage w^* such that the agent would be indifferent between taking a job or not:

$$\begin{aligned} V(w^*) &= U \\ \Rightarrow \quad w^* + \sum_{t=1}^{\infty} \beta^t w^* = \frac{w^*}{1-\beta} &= b + \beta E[J] \\ \Rightarrow \quad w^* &= (1-\beta)(b + \beta E[J]) \\ \text{And} \\ \Rightarrow \quad \beta E[J] &= \frac{w^*}{1-\beta} - b \end{aligned}$$

So we can rewrite the job value function as:

$$J(w) = \max\left\{\frac{w}{1-\beta}, \frac{w^*}{1-\beta}\right\} = \frac{1}{1-\beta}\max\{w, w^*\}$$

At the reservation wage w^*, we have

$$\begin{aligned} \frac{w^*}{1-\beta} &= b + \beta \cdot \frac{1}{1-\beta} E[\max\{w, w^*\}] \\ \Rightarrow \quad \frac{w^*}{1-\beta} - \beta\frac{w^*}{1-\beta} &= b + \frac{\beta}{1-\beta} E[\max\{w, w^*\}] - \beta\frac{w^*}{1-\beta} \\ \Rightarrow \quad w^* &= b + \frac{\beta}{1-\beta} E[\max\{w - w^*, 0\}] \\ \Rightarrow \quad w^* &= b + \frac{\beta}{1-\beta} \int_{w^*}^{\widehat{w}} w - w^* dF(w) \end{aligned}$$

Notice that the reservation wage equation is an implicit function since w^* is in the integral itself. We can thus utilize the implicit function theorem to analyze what would happen if we change β or b. Define the implicit function G as:

$$G(w^*, b, \beta) = w^* - b - \frac{\beta}{1-\beta} \int_{w^*}^{\widehat{w}} w - w^* dF(w) = 0$$

Then we have

$$\frac{\partial G}{\partial w^*} = 1 - \frac{\beta}{1-\beta} \int_{w^*}^{\widehat{w}} -1 \, dF(w) + (\widehat{w} - w^*) \cdot 0 - 0 \cdot \frac{\partial w^*}{\partial w^*}$$

$$
\begin{aligned}
&= 1 + \frac{\beta}{1-\beta}(1 - F(w^*)) \\
\frac{\partial G}{\partial \beta} &= -\frac{(1-\beta)+\beta}{(1-\beta)^2} \int_{w^*}^{\widehat{w}} w - w^* \, dF(w) \\
\frac{\partial G}{\partial b} &= -1
\end{aligned}
$$

So we have

$$
\begin{aligned}
\frac{d}{d\beta} w^* &= -\frac{-\frac{(1-\beta)+\beta}{(1-\beta)^2} \int_{w^*}^{\widehat{w}} w - w^* \, dF(w)}{1 + \frac{\beta}{1-\beta}(1 - F(w^*))} > 0 \\
\frac{d}{db} w^* &= -\frac{-1}{1 + \frac{\beta}{1-\beta}(1 - F(w^*))} > 0
\end{aligned}
$$

4.1.2 Job Search with Quitting

Consider the same model, but the worker can quit the job, wait one period and then make a decision on new job. The worker's value functions can be written as:

$$
\begin{aligned}
J(w_t) &= \max\{w_t + \beta V(w_t), b + \beta E[J]\} \\
Q(w_t) &= \max\{b + \beta E[J], w_t + \beta Q(w_t)\}
\end{aligned}
$$

Option 1: Waiting 1 Period

Suppose that the worker has to wait 1 period before they can receive and accept a new offer. Notice that if the worker decides to work in the first place, it must have been that $w_t + \beta V(w_t) \geq b + \beta E[J]$. But that would also mean that once they have decided to work, they have $b + \beta E[J] \leq w_t + \beta Q(w_t)$. This means once the worker decides to work, they would never actually decide to quit. Hence we have the exact same result as not quitting.

Option 2: No Waiting Time

Suppose that as soon as the worker quits in period t, they can receive a new job offer with $w_t \sim F(w)$, then we can write the worker's value functions as:

$$
\begin{aligned}
J(w_t) &= \max\left\{U, V^1(w), V^2(w), \dots, \lim_{n\to\infty} V^n(w_t)\right\} \\
U &= b + \beta E[J]
\end{aligned}
$$

$$V^t(w_t) = w_t + \beta w_t + \beta^2 w_t + \cdots + \beta^t E[J] + \underbrace{\left[(\beta^t w_t + \beta^{t+1} w_t + \cdots) - (\beta^t w_t + \beta^{t+1} w_t + \cdots)\right]}_{=0}$$

$$= \frac{w_t}{1-\beta} + \beta^t E[J] - \frac{\beta^t w_t}{1-\beta}$$

with

$$\begin{array}{ccc} U & \equiv & b + \beta E[J] \equiv \text{Reject Offer} \\ V^1(w) & \equiv & w_t + \beta U \equiv \text{Quit after 1 period} \\ V^2(w) & \equiv & w_t + \beta w_t + \beta^2 U \equiv \text{Quit after 2 period} \\ \vdots & & \vdots \\ V^\infty(w) & \equiv & \text{Never quits} \end{array}$$

Notice that $V^t(w)$ is increasing (decreasing) in t if $(1-\beta)E[J] < w_t$ $((1-\beta)E[J] > w_t)$. This means that the worker would only need to compare between $V^\infty(w_t), V^1(w_t)$, and U, so we can simplify the value function $J(w_t)$ as:

$$J(w_t) = \max\{U, V^1(w_t), V^\infty(w_t)\}$$

and we know that

$$U = b + \beta E[J]$$
$$V^1(w_t) = w_t + \beta E[J]$$
$$V^\infty(w_t) = \frac{w_t}{1-\beta}$$

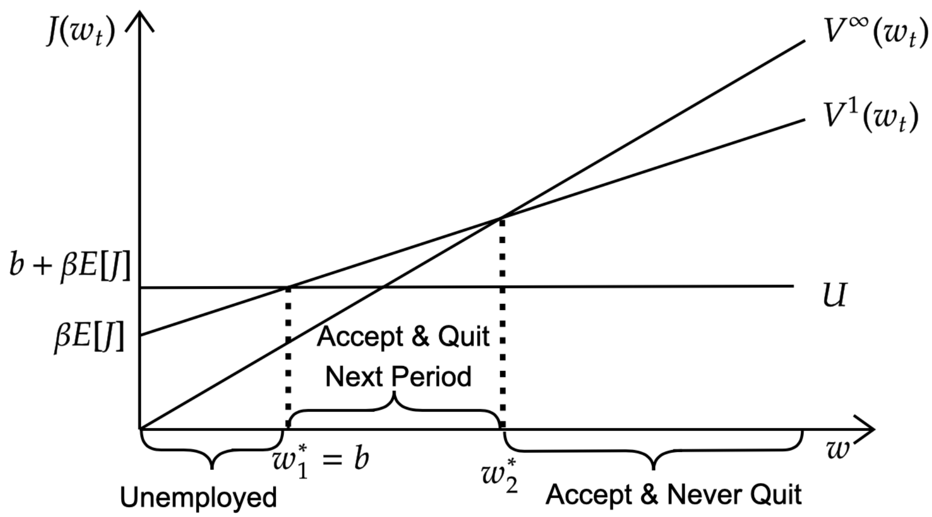
So we have 2 reservation wages w_1^*, w_2^* such that either

$$U = V^1(w^*) = w_1^* + \beta E[J]$$

or

$$V^1(w_1^*) = V^\infty(w_2^*) = \frac{w_2^*}{1-\beta}$$

As an exercise, try to solve for w_2^* using $w_2^* = (1-\beta)E[J]$. One should see that the relationship between the value functions are as depicted:



4.1.3 Job Search with Layoffs

Consider the our basic search model, but now each period the agent gets laid off with probability λ and the agent is risk-averse. Suppose that once the agent is laid off in period t, they must spend the period searching for jobs and will only receive the next job offer in period $t+1$.

We can thus write out the agent's value functions:

$$\begin{aligned} V(w_t) &= u(w_t) + \beta[(1-\lambda)V(w_t) + \lambda U] \;\Rightarrow\; V(w_t) = \frac{u(w_t) + \beta\lambda U}{1-\beta(1-\lambda)} \\ U &= b + \beta E[J] \\ J(w_t) &= \max\{V(w_t), U\} \end{aligned}$$

At the reservation wage w^* we must then have:

$$\begin{aligned} V(w^*) &= U \\ \Rightarrow\; \frac{u(w^*) + \beta\lambda U}{1-\beta(1-\lambda)} &= U \\ \Rightarrow\; u(w^*) + \beta\lambda U &= U - \beta U + \beta\lambda U \\ \Rightarrow\; u(w^*) &= (1-\beta)U = (1-\beta)(u(b) + \beta E[J]) \end{aligned} \tag{1}$$

So we can rewrite the Job value function as:

$$J(w_t) = \max\left\{\frac{u(w_t) + \beta\lambda U}{1-\beta(1-\lambda)}, \frac{u(w^*) + \beta\lambda U}{1-\beta(1-\lambda)}\right\}$$

Doing some algebra on this we get:

$$
\begin{aligned}
E[J] &= \frac{1}{1-\beta(1-\lambda)} E[\max\{u(w_t), u(w^*)\}] + \frac{\beta\lambda U}{1-\beta(1-\lambda)} \\
&= \frac{1}{1-\beta(1-\lambda)} E[\max\{u(w_t), u(w^*)\}] + \frac{\beta\lambda U}{1-\beta(1-\lambda)} - \frac{u(w^*)}{1-\beta(1-\lambda)} + \frac{u(w^*)}{1-\beta(1-\lambda)} \\
&= \frac{1}{1-\beta(1-\lambda)} E[\max\{u(w_t) - u(w^*), 0\}] + \frac{\overbrace{u(w^*) + \beta U}^{=U \text{ by equation (1)}} - \beta(1-\lambda)U}{1-\beta(1-\lambda)} \\
&= \frac{1}{1-\beta(1-\lambda)} E[\max\{u(w_t) - u(w^*), 0\}] + U
\end{aligned}
$$

From equation (1), we have:

$$
\begin{aligned}
u(w^*) &= (1-\beta)(u(b) + \beta E[J]) \\
&= (1-\beta)u(b) + \beta(1-\beta)\frac{1}{1-\beta(1-\lambda)} \int_{w^*}^{\widehat{w}} u(w) - u(w^*)\, dF(w) + \beta(1-\beta)U \\
&= (1-\beta)u(b) + \beta(1-\beta)\frac{1}{1-\beta(1-\lambda)} \int_{w^*}^{\widehat{w}} u(w) - u(w^*)\, dF(w) + \beta u(w^*) \\
\Rightarrow \quad u(w^*) &= u(b) + \frac{\beta(1-\beta)}{1-\beta(1-\lambda)} \int_{w^*}^{\widehat{w}} u(w) - u(w^*)\, dF(w)
\end{aligned}
$$

Using the implicit function theorem and Leibniz's Rule, we can show that:

$$
\frac{d}{d\lambda} w^* < 0, \ \frac{d}{d\beta} w^* > 0
$$

Using the exact same method, we can write this out for death as well.

4.1.4 Uncertainty in Offer

Consider our basic search model, but the agent only receive offers with probability α. The agent's value functions are:

$$
\begin{aligned}
J(w_t) &= \max\{V(w_t), U\} \\
V(w_t) &= \frac{w_t}{1-\beta} \\
U &= u(b) + \beta[\alpha E[J] + (1-\alpha)U]
\end{aligned}
$$

4.1.5 Multiple Offers per Period

Consider the model with layoffs, but now the agent receives N offers when they are unemployed. The wage that matters at decision time period t is thus $w_t = \max\{w_t^1, \dots, w_t^N\}$. We can thus write out the reservation wage equation as:

$$u(w^*) = u(b) + \frac{\beta(1-\beta)}{1-\beta(1-\lambda)} \int_{w^*}^{\widehat{w}} u(w) - u(w^*)\, d\tilde{F}(w)$$

where

$$\tilde{F}(w) = P(\max\{w_t^1, \dots, w_t^N\} \leq w) = P^N(w_t^j \leq w) = F^N(w)$$

4.2 Full Models with Equilibria

4.2.0 End of Period Discounting

Going forward, we will also denote discounting a little differently as

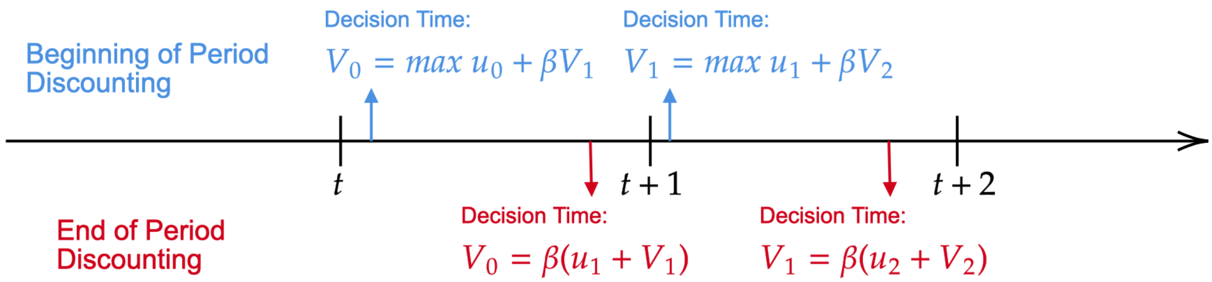
$$\beta = \frac{1}{1+r}$$

The idea of **end of period discounting (EOPD)** is simple - What if instead of making a decision to maximize present and future values, we only maximize future values and let today's value be determined yesterday?
Mathematically, in an infinite horizon model, they are practically equivalent in the sense of solving the dynamic optimization problem:

$$\begin{aligned}
\text{EOPD: } V_0^E &= \max\{\beta u_1 + \beta E[V_1^E]\} = \max\{\beta u_1 + \beta \max\{\beta u_2 + \beta E[V_2^E]\}\} \\
&= \max \sum_{t=1}^{\infty} \beta^t u_t + \lim_{t\to\infty} \beta^t E[V_t^E] \\
\text{BOPD: } V_0^B &= \max\{u_0 + \beta E[V_1^B]\} = \max\{u_0 + \beta \max\{u_1 + \beta^2 E[V_2^B]\}\} \\
&= \max \sum_{t=0}^{\infty} \beta^t u_t + \lim_{t\to\infty} \beta^t E[V_t^B]
\end{aligned}$$

So we have:

$$V_0^E = \beta V_0^B$$



4.2.1 Diamond (1982), Aggregate Demand Management in Search Equilibrium

For this model, we will operate in the following environment:

- The, measure 1, representative agent's problem is:

$$\max\ E\left[\sum_{t=0}^{\infty} \beta^t (u - c_t)\right]$$

  where c_t is the cost of production and agents derive utility u from consumption

- For this model, we will assume that agents cannot consume what they produce, rather, they have to trade to consume
- There are 2 sectors in this economy: Production ($1 - N_t$ agents) and Trade (N_t agents)
- α_0 (α_1) is the probability of finding a production (trade) opportunity
- γ_0 (γ_1) is the search cost per period in the production (trade) sector
- The cost of production is a random variable $c_t \sim F(c)$

We can set up the agent's value function as:

$$V_0 = \frac{1}{1+r}\Bigg\{ -\gamma_0 + \underbrace{\alpha_0 E[\max\{V_1 - c, V_0\}]}_{\substack{\text{Find an opportunity} \\ \text{and then decide} \\ \text{whether to produce}}} + \underbrace{(1-\alpha_0)V_0}_{\text{Found no opportunity}} \Bigg\} \tag{1}$$

$$V_1 = \frac{1}{1+r}\Bigg\{ -\gamma_1 + \underbrace{\alpha_1(u + V_0)}_{\substack{\text{Find an opportunity, consume,} \\ \text{and go back to production}}} + \underbrace{(1-\alpha_1)V_1}_{\text{Found no opportunity}} \Bigg\} \tag{2}$$

Multiply equations (1) and (2) by $(1 + r)$ and subtract V_0 and V_1, respectively, we get the flow equations[12]:

$$rV_0 = -\gamma_0 + \alpha_0 E[\max\{V_1 - V_0 - c, 0\}] \tag{3}$$
$$rV_1 = -\gamma_1 + \alpha_1(u + V_0 - V_1) \tag{4}$$

For simplicity, let's assume that $\gamma_0 = \gamma_1$, subtracting equation (3) from equation (4), we get:

$$r(V_1 - V_0) = \alpha_1 u - \alpha_1(V_1 - V_0) - \alpha_0 E\left[\max\{V_1 - V_0 - c, 0\}\right] \tag{5}$$

Define $R \equiv V_1 - V_0$ to be the reservation "cost" of producing for subsequent trade. Assume that R is constant over time (at least in equilibrium), then we can rewrite equation (5) as:

$$\begin{aligned} rR &= \alpha_1(u - R) - \alpha_0 E[\max\{R - c, 0\}] \\ &= \alpha_1(u - R) - \alpha_0 \int_{\underline{c}}^{R} -c + R \, dF(c) \end{aligned} \tag{6}$$

where $\underline{c} = \inf\{supp(F(c))\}$. As such, the amount of people in the trade sector in period $t+1$ is:

$$N_{t+1} = \underbrace{N_t}_{\substack{\text{Number of people} \\ \text{in trade sector in } t}} + \underbrace{(1 - N_t)\alpha_0 F(R)}_{\substack{\text{Number of people who decided to} \\ \text{produce for subsequent trade in } t}} - \underbrace{\alpha_1 N_t}_{\substack{\text{Number of people who} \\ \text{were able to trade in } t}}$$

So in steady-state, we should have

$$\begin{aligned} N &= N + (1 - N)\alpha_0 F(R) - \alpha_1 N \\ \Rightarrow \quad N &= \frac{\alpha_0 F(R)}{\alpha_1 + \alpha_0 F(R)} \end{aligned} \tag{7}$$

To study the locus of (R, N), we can define implicit functions from equations (6) and (7):

$$S(R, N) \equiv rR - \alpha_1(u - R) + \alpha_0 \int_{\underline{c}}^{R} -c + R \, dF(c)$$
$$T(R, N) \equiv N\alpha_1 - (1 - N)\alpha_0 F(R)$$

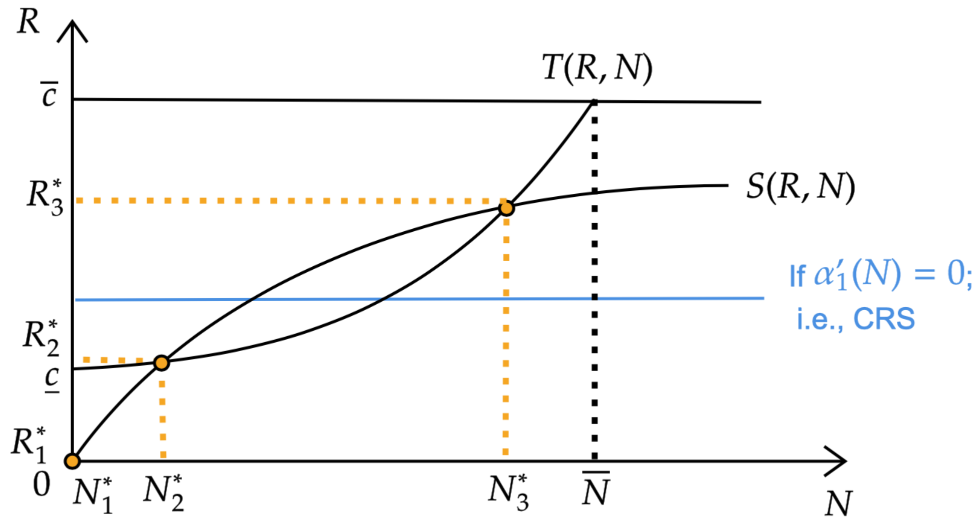
We can thus characterize the steady-state equilibrium as (R, N) such that:

[12] I don't really know why they are called flow equations. If you know, please tell me.

(1) $S(R, N) = 0$

(2) $T(R, N) = 0$

In this case, we actually have 3 equilibria: $(0, 0)$ aka Autarky, (R_2^*, N_2^*), and (R_3^*, N_3^*)



Exercise: Show that $S(R, N)$ is increasing. We can think of the probability of meeting a trade opportunity as a function of N:

$$\alpha_1(N) = \frac{\#\text{ of meetings}}{N}$$

If we assume that $\alpha_1'(N) > 0$ (meaning increasing return to scale and "thick market externality").

Welfare: For the social planner, the whole economy's welfare is:

$$W = (1 - N)V_0 + NV_1$$
$$\Rightarrow \; rW = (1 - N)rV_0 + NrV_1$$

We can actually show that $W_3 > W_2$

The Diamond Paradox:

So far, we have been assuming that the job offer that a worker receives in period t is a random variable w_t drawn from the distribution $F(w)$. But if we can actually solve for a reservation wage, what even is the point of such distribution? Why would firms not just offer reservation wage only?

To answer this, Diamond showed that, if there is no immediate unemployment benefits, there will be no reservation wage.

Suppose otherwise that a reservation wage $\tilde{R}$ exists. Then, at that wage, for a worker to keep working, it must be that, for some $\varepsilon > 0$:

$$V_w(\tilde{R}) = \underbrace{\frac{\tilde{R}-\varepsilon}{1-\beta}}_{\substack{\text{Value function of accepting a} \\ \text{slightly lower offer in period } t}} \geq \underbrace{\beta\frac{\tilde{R}}{1-\beta}}_{\substack{\text{Value function of no work in } t \\ \text{but accept job in } t+1}}$$

4.2.2 Kiyotaki & Wright (1993), Search Theoretic Approach to Monetary Economics

For this model, we will operate in the following environment:

- There are 2 types of agents in this economy. $M \in [0, 1]$ agents with one unit of money, and $1 - M$ agents without money
- All agents derive utility u from owning a good
- α is the probability of meeting another agent
- x is the probability that one agent owns a good that the other agent wants
- The cost of production is a 0
- States: $\{0 \equiv$ seller (no money), $1 \equiv$ buyer (has money)$\}$
- π is the strategy of whether a seller accepts money as payment for good or not[13]
- Π is the buyers' belief about the probability that the seller will accept money for the good

[13]It turns out to be quite important to think of this as a strategy (i.e., maximizer of utility) and not a probability. Sang Joon Rhee and I had a 2-hour discussion on the exact composition of V_0 because we thought it was a probability.

We can then set up the agents' value functions:

$$V_0 = \frac{1}{1+r}\Bigg\{ \underbrace{\alpha(1-M)x^2(u+V_0)}_{\substack{\text{Seller meets another seller and}\\ \text{they want each other's good}}} + \underbrace{\alpha Mx\max\{V_1,V_0\}}_{\substack{\text{Seller meets a buyer and}\\ \text{buyer wants to buy}\\ \text{Seller may or may not sell}}} + \underbrace{\left[1-\alpha(1-M)x^2-\alpha Mx\pi\right]V_0}_{\text{Everything else}} \Bigg\} \tag{1}$$

$$V_1 = \frac{1}{1+r}\Bigg\{ \underbrace{\alpha(1-M)x\Pi(u+V_0)}_{\substack{\text{Buyer meets seller and}\\ \text{wants seller's good}}} + \underbrace{[1-\alpha(1-M)x\Pi]V_1}_{\text{Buyer no buy}} \Bigg\} \tag{2}$$

Just like in the Diamond model, we will multiply equations (1) and (2) by (1 + r) and subtract V_0 and V_1, respectively. Then, we get the flow equations:

$$rV_0 = \alpha(1-M)x^2u + \alpha Mx\max\{V_1-V_0,0\} \tag{3}$$

$$rV_1 = \alpha(1-M)x\Pi(u+V_0-V_1) \tag{4}$$

Notice that, in equation (1), we can characterize the maximization problem with the seller strategy[14] π:

$$\max\{V_1-V_0,0\} = \max_{\pi\in[0,1]} \pi(V_1-V_0) \;\Rightarrow\; \pi = \begin{cases} 1 & \text{if } V_1-V_0>0 \\ \varphi & \text{if } V_1-V_0=0 \\ 0 & \text{if } V_1-V_0<0 \end{cases}$$

Using this fact, we can write equation (4) minus (3) as:

$$r(V_1-V_0) = \alpha(1-M)x(\Pi-x)u - \alpha(1-M)x\Pi(V_1-V_0) - \alpha Mx\pi(V_1-V_0)$$

$$\Rightarrow\; V_1-V_0 = \frac{\alpha(1-M)x(\Pi-x)u}{r+\alpha(1-M)x\Pi+\alpha Mx\pi}$$

Since the denominator is strictly positive, the relationship between Π and x determines the sign of V_1-V_0, so we can rewrite π as a function of Π:

$$\pi(\Pi) = \begin{cases} 1 & \text{if } \Pi-x>0 \\ \varphi & \text{if } \Pi-x=0 \\ 0 & \text{if } \Pi-x<0 \end{cases}$$

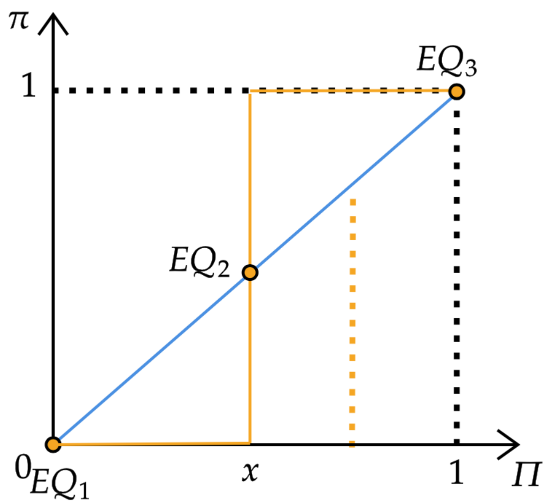
[14] Sometimes people think of this as a probability, but really you need to think about it as a strategy. If $\pi=\varphi$, then it's a mixed-strategy of randomly accepting money.

In this case, we actually have 3 equilibria:

EQ_1- No Money Accepted in Trade: $\pi = \Pi = 0$

EQ_2- Some Money Accepted in Trade: $\pi = \Pi = x$

EQ_3- All Money Accepted in Trade: $\pi = \Pi = 1$



Welfare: For the social planner, the whole economy's welfare is:

$$W = (1 - M)V_0 + MV_1 = V_0 + M(V_1 - V_0)$$

So we can make welfare comparisons between each equilibrium:
In the **first equilibrium** (no money accepted in trade), we have:

$$\begin{aligned} rW &= r(1-M)V_0 + MrV_1 = rV_0 + Mr(V_1 - V_0) \\ &= \underbrace{\alpha(1-M)x^2u + 0}_{rV_0|_{\pi=0}} + M[\underbrace{-\alpha(1-M)x^2u - 0 - 0}_{r(V_1-V_0)|_{\Pi=0}}] \\ &= \alpha(1-M)^2x^2u \\ M^* &\equiv \underset{M\in[0,1]}{argmax}\ rW = 0 \end{aligned}$$

In the **second equilibrium** (some money accepted in trade), we have:

$$\begin{aligned} rW &= \underbrace{\alpha(1-M)x^2u + 0}_{rV_0|_{V_1-V_0=0}} + M[\underbrace{0}_{r(V_1-V_0)|_{\Pi=x}}] \\ &= \alpha(1-M)x^2u \\ M^* &\equiv \underset{M\in[0,1]}{argmax}\ rW = 0 \end{aligned}$$

In the **third equilibrium** (all money accepted in trade, i.e., $\pi = 1$), we have:

$$
\begin{aligned}
rW &= \underbrace{\alpha(1-M)x^2u + \alpha Mx(V_1 - V_0)}_{rV_0|_{V_1-V_0>0}} + Mr(V_1 - V_0) \\
&= \alpha(1-M)x^2u + \alpha Mx\frac{\alpha(1-M)x(1-x)u}{r+\alpha(1-M)x+\alpha Mx} + Mr\cdot\frac{\alpha(1-M)x(1-x)u}{r+\alpha(1-M)x+\alpha Mx} \\
&= \alpha(1-M)x^2u + M(ax+r)\frac{\alpha(1-M)x(1-x)u}{r+\alpha(1-M)x+\alpha Mx} \\
&= \alpha(1-M)x^2u + M\cancel{(ax+r)}\frac{\alpha(1-M)x(1-x)u}{\cancel{r+\alpha x}} \\
&= \alpha u[x^2 - Mx^2 + Mx - M^2x - Mx^2 + M^2x^2] \\
&= \alpha u[x^2 - 2Mx^2 + Mx - M^2x + M^2x^2] \\
M^* &\equiv \underset{M\in[0,1]}{argmax}\ rW = \frac{2x-1}{2x-2}
\end{aligned}
$$

This means that $x \downarrow \Rightarrow\ M^* \uparrow$ (money for goods trade $\uparrow$).

4.2.3 Burdett & Wright (1998), Two-Sided Search with Non-Transferable Utility

For this model, we will operate in the following environment:

- There are 2 types of agents $i \in \{w, e\}$ in this economy, workers (w) and employers (e)
- If a worker is employed, they receive wage $z_w \sim F_w(z)$
- When an employer hires a worker, the worker provides productivity $z_e \sim F_e(z)$
- α_i is the probability of receiving/giving a job offer
- δ_i is the probability of death/bankruptcy
- λ_i is the probability of being laid-off/being quitted on
- c_i is the benefit of unemployement/no-hire
- For this model, we will assume that agents have a linear utility function

We can then set up the value functions for the agents:

$$V_w(z) = \frac{1-\delta_w}{1+r_w}\Big\{z_w + \lambda_w U_w + (1-\lambda_w)V_w(z_w)\Big\} \tag{1}$$

$$U_w = \frac{1-\delta_w}{1+r_w}\left\{c_w + \alpha_w E[\max\{V_w(z), U_w\}] + (1-\alpha_w)U_w\right\} \tag{2}$$

Let's do something a little sketchy. Assume that δ_w is sufficiently small, such that

$$\frac{1}{1-\delta_w} \approx 1 + \delta_w$$

Then, like when we did the Fisher equivalence back in 813A, we need to assume that because δ_w is so small, $\delta_w \cdot r_w \approx 0$.
Under these assumptions, we have:

$$\frac{1-\delta_w}{1+r_w} \approx \frac{1}{(1+\delta_w)(1+r_w)} \approx \frac{1}{1+\delta_w+r_w}$$

Just like in the last two models, we will multiply equations (1) and (2) by $(r_w + \delta_w)$ and subtract V_w and U_w, respectively. We get the flow equations:

$$(r_w + \delta_w)V_w(z_w) = z_w + \lambda_w[U_w - V_w(z_w)] \tag{3}$$

$$(r_w + \delta_w)U_w = c_w + \alpha_w E[\max\{V_w(z_w) - U_w, 0\}] \tag{4}$$

Rewriting equation (3), we get

$$V_w(z_w) = \frac{z_w + \lambda_w U_w}{r_w + \delta_w + \lambda_w}$$

Next, let R_w be the worker's reservation wage. At $z_w = R_w$, we should get $V_w(R_w) = U_w$, so $R_w = (r_w + \delta_w)U_w$[15]. Subtract $(r_w + \delta_w)U_w = R_w$ from equation (3), we get:

$$(r_w + \delta_w)[V_w(z_w) - U_w] = z_w + \lambda_w[U_w - V_w(z_w)] - R_w$$

which gives us:

$$V_w(z_w) - U_w = \frac{z_w - R_w}{r_w + \delta_w + \lambda_w}$$

[15] Because $(r_w + \delta_w)V_w(R_w) = z_W + \lambda_w \cdot 0 = (r_w + \delta_w)U_w$. We will use this for Eq. (5)

Recall from equation (4) that $R_w = (r_w + \delta_w)U_w = c_w + \alpha_w E[\max\{V_w(z) - U_w, 0\}]$, we can thus write

$$R_w = (r_w + \delta_w)U_w = c_w + \alpha_w \int_{R_w}^{\hat{z}} [V_w(z_w) - U_w]\, dF_w(z) \tag{5}$$

$$= c_w + \alpha_w \int_{R_w}^{\hat{z}} \frac{z - R_w}{r_w + \delta_w + \lambda_w}\, dF_w(z)$$

$$= c_w + \frac{\alpha_w}{r_w + \delta_w + \lambda_w} \int_{R_w}^{\hat{z}} z - R_w\, dF_w(z) \tag{6}$$

By symmetry, employers face the same problem and will have the same *reservation productivity* as equation (6):

$$R_e = c_e + \frac{\alpha_e}{r_e + \delta_e + \lambda_e} \int_{R_e}^{\hat{z}} z - R_e\, dF_e(z) \tag{7}$$

Now, let's throw in some search frictions. Suppose that workers/employers can only find a job/new-hire with probability[16] β_i. We can calculate beta as:

$$\alpha_w = \beta_w \cdot Prob(z > R_w) = \beta_w[1 - F_e(R_e)]$$
$$\alpha_e = \beta_e \cdot Prob(z > R_e) = \beta_e[1 - F_w(R_w)]$$

With these equations, we can now qualify the equilibria in this economy. Any equilibrium in this economy should be a pair (R_w, R_e) such that:

(1) $R_e(R_w)$ and $R_w(R_e)$ satisfy equations (6) and (7)

(2) $\alpha_e = \beta_e \cdot [1 - F_w(R_w)]$, $\alpha_w = \beta_w \cdot [1 - F_e(R_e)]$ where β_e and β_w are exogenously given

(3) $\lambda_w = \delta_e$, $\lambda_e = \delta_w$ (Work until worker dies or employer bankrupts, and then start searching for new opportunity)

[16] Note that this is different from α_i, the probability of getting an offer. α_w involves (1) P(Worker finds firm)= β_i (2) P(worker productivity is above reservation productivity)

Replacing the α_i's and λ_i's, we can rewrite equations (6) and (7) as:

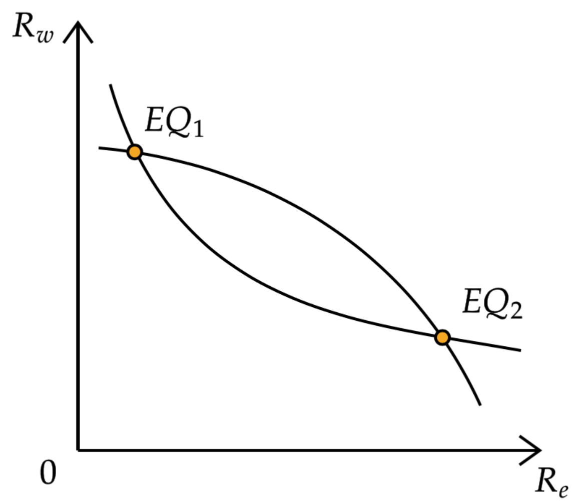
$$R_w(R_e) = c_w + \frac{\beta_w \cdot [1 - F_e(R_e)]}{r_w + \delta_w + \delta_e} \int_{R_w}^{\hat{z}} z - R_w \, dF_w(z)$$

$$R_e(R_w) = c_e + \frac{\beta_e \cdot [1 - F_w(R_w)]}{r_e + \delta_e + \delta_w} \int_{R_e}^{\hat{z}} z - R_e \, dF_e(z)$$

Without showing any work, this graph represents what the equilibria should look like

EQ_1- Workers are picky and firms would take just about anyone

EQ_2- Firms are picky and workers would take just about job offer



In this scenario, we can have multiple equilibria driven by self-fulling expectations. If workers is could have been picky but they "thought" β_w is very low, then they can slowly move towards EQ_2, even if they were originally closer to EQ_1.

Exercise: Show that $\frac{\partial R_w}{\partial R_e} < 0$ and $\frac{\partial R_e}{\partial R_w} < 0$

5 Growth Models (Exogenous)

To study growth models, we need first to understand what we are even trying to study. Consider an economy where the GDP can be described by:

$$Y_t = z_t Y_0 e^{g(t)} = \underbrace{\tau_t}_{Trend} + \underbrace{d_t}_{Deviation/Noise}$$

Such a model is simplistic, but it provides a decent thought experiment for starting some basic modeling. But is there a way we can truly separate the progression of our economy into trend and noise? If so, can we simply study τ_t to understand the growth of an economy and treat business cycles as deviation from trend?

The **Hodrick-Prescott Filter** attempts to answer this question for us. First, let's think about what "trend" represents. If there truly was a trend, our best approximation/estimation of it is probably through finding the minimizer of the noise[17]. Formally, this means we want to find τ_t that solves the problem:

$$\underset{\tau_t}{argmin} \sum_{t=1}^{T} (Y_t - \tau_t)^2 \ s.t. \ \sum_{t=2}^{T-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \leq \mu$$

The minimization part is set up like an OLS problem as that calculates for the linear minimizer. The constraint part is set up so that the τ_t's are truly the trend so that changes from $t-1$ to t to $t+1$ are restricted to be pretty small. In empirics, the constraint μ is chosen to be around 1600.

If we are successful in identifying the trends, how can we interpret their value? Is a 2% annual growth of GDP is very different from an 8% annual growth[18]?

Obviously, this is not the end-all-be-all of Macroeconomics. In this chapter, we will learn more about different ways to model growth, along with all the math we need to understand and expand on them.

[17]We can think of it as writing GDP as a conditional mean of the trend so that we study $E[Y_t \mid \tau_t]$

[18]In growth, there is a thing called the 72 rule. Take 72 and divide it by the percentage point growth rate, and you will get the number of periods needed to double the original amount. E.g., A 2% annual growth rate doubles the GDP in $\frac{72}{2} = 36$ years. An 8% annual growth rate doubles the GDP in $\frac{72}{8} = 9$ years. So these two are obviously very very different.

5.1 Solow Growth Model

The first thing we will learn is the infamous *Solow Growth Model*. This model is notoriously simplistic, and is often criticized for being "basically not micro-founded". We, as students of economics, should thus investigate, and hopefully, learn from both its virtues and its faults.

Like any model we have learned thus far, we will begin by defining the **Environment**:

- We will describe the economy with $Y_t = A_t F(H_t, K_t)$ where H_t is labor, K_t is capital stock, and A_t is an exogenous (or potentially endogenous) source of the business cycle.
- $F(H_t, K_t)$ is assumed to be a **Constant Rate of Techinical Substituion (CRTS)** production function.
- Out time will be discrete $t = 1, \ldots, n \in \mathbb{N}$
- Closed economy $S_t = I_t$ (Saving through investments only. No bonds.). We can thus write $S_t = sY_t$ where s is an endogenous variable that maximizes the present value utility function.
- $K_{t+1} = K_t(1-\delta) + I_t$ where δ is the depreciation rate and I_t is investment
- $\dfrac{H_{t+1}}{H_t} = 1 + n$ is the population growth, where n is the exogenous growth rate
- Define GDP and capital percapita as $y_t \equiv \dfrac{Y_{t+1}}{H_t}$, $k_t \equiv \dfrac{K_{t+1}}{H_t}$. In our closed economy, our main interest is to study k_t, the only source of variation in this simplistic world.

Notice that since $F(H, K)$ is *CRTS*, we have

$$
\begin{aligned}
& K_{t+1} = K_t(1-\delta) + I_t = K_t(1-\delta) + sA_tF(H_t, K_t) \\
\Rightarrow \quad & \underbrace{\frac{K_{t+1}}{\cancel{H_{t+1}}}}_{=k_{t+1}} \frac{\cancel{H_{t+1}}}{H_t} = \underbrace{\frac{K_t}{H_t}}_{=k_t}(1-\delta) + sA_t \underbrace{F(1, \frac{K_t}{H_t})}_{\text{Call this } f(k_t)} \\
\Rightarrow \quad & k_{t+1}(1+n) = (1-\delta)k_t + sA_t f(k_t)
\end{aligned}
$$

By this point of the semester, it should be intuitive for you to want to find/define the steady-state once we have our model set up. Can we find a steady-state in this environment?

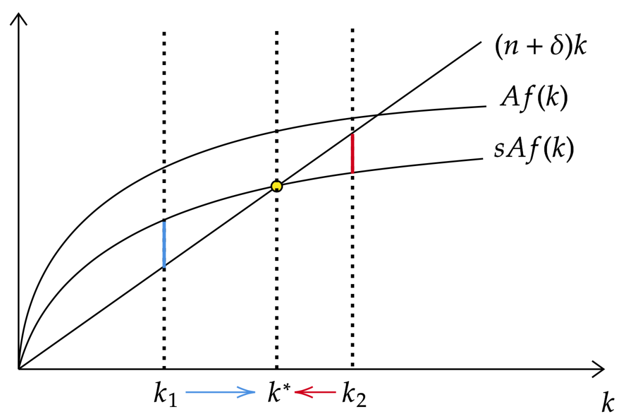
Suppose so, then in steady-state we must have $k_t = k_{t+1} = k^*$, meaning that

$$(1+n)k^* = (1-\delta)k^* + sAf(k^*) \quad \Rightarrow \quad (n+\delta)k^* = sAf(k^*)$$

What does this equation mean?
LHS $(n+\delta)k^*$ is the increase in capital in the economy to keep the capital per captia constant[19].
RHS $sAf(k^*)$ is the investment needed so that $k_t = k^*$ and the economy's growth in capital can keep up with the depreciation and the growth in population.

Recall our fairly reasonable assumption that $\lim_{k\to 0} f'(k) = \infty$ (marginal product of capital tends to infinity as capital tends to 0). Given our assumption that $f(k)$ is concave, the graph below illustrates what the steady-state equilibrium should look like:

Notice that to the left of the yellow dot (say k_1), the economy is saving $(sAf(k_1))$ more than necessary to keep up with $(n+\delta)k_1$.
To the right of the yellow dot (say k_2), the economy is saving $(sAf(k_1))$ less than necessary to keep up with $(n+\delta)k_1$.
As such, both k_1 and k_2 would grow towards k^* and so k^* is a stable equilibrium.



If n or δ increases, k^* will decrease. If s (endogenously) increases, k^* will increase (note that this does not imply that agents are better off). We can also study what happens if we assume that A is constant, or we can condition on s or δ and so on...you get the idea.

How would this model fair in **continuous time**?
For continuous time, we will use "dot" to denote a derivative with respect to time. For example, $\frac{\partial}{\partial t}K = \dot{K}$

$$\begin{aligned}
\dot{K} &= I - \delta K \\
\dot{k} &= \left(\frac{\dot{K}}{H}\right) = \frac{\dot{K}H - K\dot{H}}{H^2} = \frac{\dot{K}}{H} - k\underbrace{n}_{\equiv\frac{\dot{H}}{H}} \\
&= \frac{sAF(H,K) - \delta K}{H} - kn \\
\dot{k} &= sAf(k) - (n+\delta)k
\end{aligned}$$

[19]population growth rate plus the depreciation rate and then multiply the capital (and then divide by population).

So exactly the same as discrete time, the steady-state condition is described by:

$$\dot{k} = 0 \ \Rightarrow \ sAf(k^*) = (n+\delta)k^*$$

Since s, A, n, δ are all just rates, the natural next step is thus to discuss the model under different production technologies.

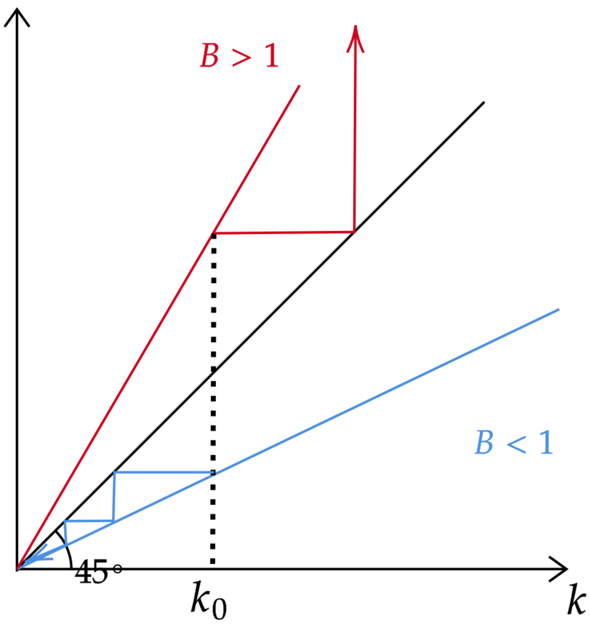
5.1.1 Linear Technology

Testing the water with this simplistic case. We can qualify the steady-state equilibria easily:

Environment:

- $f(k) = Ak$ (constant returns to scale)
- $n = 0$
- $k_{t+1} = k_t(1-\delta) + sAk_t$
  $= \underbrace{(1-\delta+sA)}_{\text{Call this } B} k_t.$
  So
  $k_{t+1} = Bk_t = B^2 k_{t-1} = \cdots = B^{t+1}k_0.$

Notice this means that a necessary (but not sufficient) condition of steady-state is thus $B \in [-1, 1]$



5.1.2 Labor-Augmenting Technological Process

For our **Environment**, we shall assume that

- $Y_t = AF(H\gamma^t, K_t)$, $\gamma > 1$ (workers become more productve over time by rate of γ)
- $n = 0 \ \Rightarrow \ H$ is constant
- Cobb-Douglas production function: $Y = AK^\alpha H^{1-\alpha} \ \Rightarrow \ y = \dfrac{Y}{H} = Ak^\alpha$

Recall that $k_{t+1} = k_t(1-\delta) + sAF(\gamma^t, k_t)$. Let $\widehat{k_t} = \gamma^{-t}k_t$, we have:

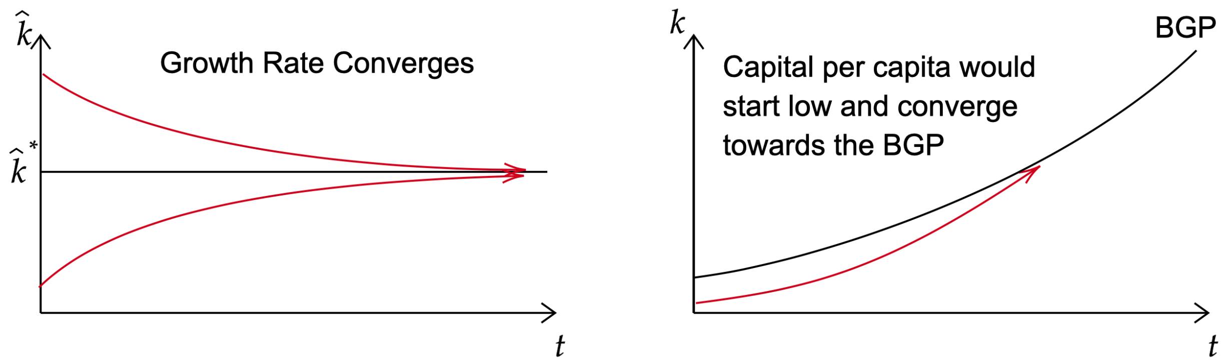
$$\frac{k_{t+1}}{\gamma^t} = \frac{k_t}{\gamma^t}(1-\delta) + sAf(\frac{k_t}{\gamma^t})$$

$$\Rightarrow \gamma \widehat{k_{t+1}} = \widehat{k}_t(1-\delta^1) + s^1 A f(\widehat{k}_t)$$
$$\Rightarrow \widehat{k_{t+1}} = \widehat{k}_t \underbrace{(1-\widehat{\delta})}_{\frac{1-\delta}{\gamma}} + \underbrace{\widehat{s}}_{\frac{s}{\gamma}} A f(\widehat{k}_t)$$

Notice that this is in the exact same form as the general case, so we know that there exists a unique steady-state equilibrium.

Since $k_t = \gamma^t \widehat{k}^*$, over time, capital grows at rate γ in steady-state. We shall call this the **Balanced Growth Path**.

Definition (BGP): A **Balanced Growth Path** is a sequence (y_t, c_t, k_t) such that each variable grows at a constant rate. Along the BGP, $\widehat{k}^*$ is a constant. If $A_t = \gamma^t A$, then there exists a *BGP*. The figure below illustrates how convergence to BGP works.



5.2 Neoclassical Growth Model (Ramsey-Cass-Koopnan)

One may notice that the simplicity of the Solow growth model is elegant yet lacks microeconomics foundations. The *Neoclassical growth model* tries to bring in micro-foundation and change how growth is studied if agents in the economy make investment decisions while maximizing consumption.

Recall our classic social planner's problem (lower case letters will now represent the variables itself, and not per capita):

$$\max_{k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$s.t. \begin{cases} c_t = F(k_t) - k_{t+1} + k_t(1-\delta) \\ i_t = k_{t+1} - k_t(1-\delta) \end{cases}$$

This should be familiar territory to the readers, as we are back to discussing materials from 813A:

- State Variables: $\{k_t\}$
- Control Variables: $\{c_t,\, i_t,\, k_{t+1}\}$

Given that we are in a closed economy, c_t and i_t are exogenous once k_{t+1} is chosen. Hence we will focusing on studying the relationship between k_{t+1} and k_t (rings any bell?).

As such, we can write our transition equation $x_{t+1} = f(x_t, a_t, \varepsilon_t)$ in a deterministic economy:

$$\underbrace{k_{t+1}}_{\text{state at } t+1} = f(k_t, k_{t+1}, 0) = \underbrace{k_{t+1}}_{\text{choice at } t+1}$$

With that, let's set up the Bellman Equation:

$$V(k_t) = \max_{k_{t+1}} \{u(c_t) + \beta V(k_{t+1})\}$$

The F.O.C. (w.r.t. k_{t+1}) is:

$$-u'(c_t) + \beta V'(k_{t+1}) = 0 \tag{1}$$

This gives us the optimal decision rule $k_{t+1} = \alpha(k_t)$ that satisfies $u'(c_t) = \beta V'(k_{t+1})$. Now how does this rule change with respect to k_t? Differentiating equation (1) w.r.t. k_t, we get

$$-u''(c_t)[F'(k_t) - \alpha'(k_t) + (1-\delta)] + \beta V''(k_{t+1})\alpha'(k_t) = 0$$

Doing some algebra we get (recall that, as done in the homework, given concave utility function, the value function is also concave):

$$\alpha'(k_t) = \frac{\overbrace{u''(c_t)[F'(k_t) + (1-\delta)]}^{<0}}{\underbrace{u''(c_t) + \beta V''(k_{t+1})}_{<0}} > 0$$

So the optimal decision rule is an increasing function in k_t.
Now, by the envelope theorem, the optimal decision rule needs to satisfy:

$$V'(k_t) = u'(c_t)[F'(k_t) - \alpha'(k_t) + (1-\delta)] + \beta V'(k_{t+1}\alpha'(k_t)$$

$$= u'(c_t)[F'(k_t) + (1-\delta)] + \alpha'(k_t)\underbrace{[-u'(c_t) + \beta V'(k_{t+1}]}_{=0 \text{ by equation (1)}}$$
$$= u'(c_t)[F'(k_t) + (1-\delta)]$$

Updating 1 period, we get

$$V'(k_{t+1}) = u'(c_{t+1})[F'(k_{t+1}) + (1-\delta)] \tag{2}$$

Combining equations (1) and (2), we get our *Euler Equation*

$$u'(c_t) = \beta u'(c_{t+1})[F'(k_{t+1}) + (1-\delta)]$$

Dividing $\beta u'(c_{t+1})$ we get the inter-temporal marginal rate of substitution for consumption:

$$MRS = \frac{u'(c_t)}{\beta u'(c_{t+1})} = F'(k_{t+1}) + 1 - \delta = MRT \tag{3}$$

Since c_t is a function of k_t and k_{t+1} and c_{t+1} is a function of k_{t+1} and k_{t+2}, equation (3) involves 3 periods (t, $t+1$, $t+2$). As you may recall from either 813A or any other ordinary differential equation class: To pin down a solution (for $\alpha(k_t)$), we need **2** conditions:

(i) k_0 is known

(ii) Transversality Condition (TVC): $\lim_{t\to\infty} \beta^t u'(c_t)k_{t+1} = 0$

More on the Transversality condition:
Recall that, in a finite horizon economy, the social planner's problem is:

$$\max \sum_{t=0}^{T} \beta^t u(c_t) = \max \sum_{t=0}^{T} \beta^t u\left([F(k_t) - k_{t+1} + k_t(1-\delta)]\right)$$

The F.O.C. of the last control variable (k_{T+1}) is:

$$-\beta^T u'(c_T)[F'(k_T) + (1-\delta)] \leq 0$$

If the utility function is strictly increasing, then the maximized endpoint should be that $k_{T+1} = 0$, otherwise, things go to waste. But if the utility function is only weakly increasing, or $u(c_T)$ is discounted so much that any endpoint capital stock in the neighborhood around $k_{T+1} > 0$ still maximizes. So we can pin down the maximizing condition as either $\lim_{T\to\infty} \beta^T u'(c_T) = 0$ if $\lim_{T\to\infty} k_{T+1} > 0$, or just $k_{T+1} = 0$. For simplicity, we can multiply the two and set our transversality condition as

$$\lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0$$

Next, let's describe the steady-state equilibrium under this model. We know that any optimal rule $k_{t+1} = \alpha(k_t)$ must satisfy:

$$\begin{aligned} u'(c_t) &= \beta u'(c_{t+1})[F'(\alpha(k_t)) + 1 - \delta] \\ k_{t+1} &= \alpha(k_t) = F(k_t) - c_t + k_t(1-\delta) \end{aligned}$$

In the steady-state, we should have that $k_t = k_{t+1} = k^*$ and $c_t = c_{t+1} = c^*$, so we can rewrite these conditions as:

$$\begin{aligned} u'(c^*) &= \beta u'(c^*)[F'(k^*) + 1 - \delta] \\ k^* &= \alpha(k^*) = F(k^*) - c^* + k^*(1-\delta) \end{aligned}$$

Let $\beta = \frac{1}{1+\rho}$, we can rewrite the steady-state condition as:

$$\begin{aligned} \rho + \delta &= F'(k^*) \\ c^* &= F(k^*) - \underbrace{\delta k^*}_{\text{Invest exactly to offset depreciation}} \end{aligned}$$

To maximize c^* through k^*, the first order condition is that $F'(k^*) - \delta = 0$. This gives us what is called the **Golden Rule Capital Stock** k^{GR}.

Since the first steady-state condition is that $F'(k^*) = \rho + \delta > \delta = F'(k^{GR})$ and that we assume F to be increasing but strictly concave, we know that we must have $k^* = k^{SS} < k^{GR}$.

Below are some examples to show the difference between k^{SS} and k^{GR} in equilibrium.

5.2.1 Cobb-Douglas Production Function

Let $u(c) = ln(c)$ and $F(k) = k^\theta$ with capital depreciation rate $\delta = 1$. Our Bellman equation is:

$$V(k_t) = \max_{k_{t+1}} \; ln(c_t) + \beta V(k_{t+1})$$

Since $u(c) = ln(c)$ is increasing and concave on $\mathbb{R}_{++}$, we know that $V(k_t)$ is increasing and concave, and that we can define a contraction mapping such that $TV = V$.
Suppose we start with $V_1(k_{t+1}) = 0$, then $V_2(k_t) = TV_1(k_t)$ and so

$$V_2(k_t) = TV_1(k_t) = \max_{k_{t+1}} \; ln(k_t^\theta - \underbrace{k_{t+1}}_{\text{since } \delta=1}) + \beta \cdot \underbrace{0}_{V_1(k_{t+1})=0}$$

$$\Rightarrow \quad k_{t+1} = 0$$

$$\Rightarrow \quad V_2(k_t) = \theta ln(k_t)$$

We then repeat this with $V_3(k_t)$ so

$$V_3(k_t) = TV_2(k_t) = \max_{k_{t+1}} \; ln(k_t^\theta - k_{t+1}) + \beta \cdot \underbrace{\theta ln(k_{t+1})}_{V_2(k_{t+1})}$$

The F.O.C. of this maximization problem is:

$$\frac{1}{k_t^\theta - k_{t+1}} = \frac{\beta\theta}{k_{t+1}} \;\Rightarrow\; k_{t+1} = \frac{\beta\theta}{1+\beta\theta}k_t^\theta$$

$$\Rightarrow \; V_3(k_t) = ln(k_t^\theta - \frac{\beta\theta}{1+\beta\theta}k_t^\theta) + \beta\theta ln(k_{t+1})$$

We then repeat this with $V_4(k_t)$ so

$$V_4(k_t) = TV_3(k_t) = \max_{k_{t+1}} \; ln(k_t^\theta - k_{t+1}) + \beta \cdot \underbrace{\theta ln(\frac{1}{1+\beta\theta}k_{t+1}^\theta)}_{V_3(k_{t+1})}$$

The F.O.C. of this maximization problem is:

$$\frac{1}{k_t^\theta - k_{t+1}} = \frac{\beta\theta(1+\beta\theta)}{k_{t+1}} \;\Rightarrow\; k_{t+1} = \frac{\beta\theta(1+\beta\theta)}{1+\beta\theta(1+\beta\theta)}k_t^\theta$$

So

$$V_4(k_t) = ln(\frac{1}{1+\beta\theta(1+\beta\theta)}k_t^\theta) + \beta\theta(1+\beta\theta)ln\left(\frac{\beta\theta(1+\beta\theta)}{1+\beta\theta(1+\beta\theta)}k_t^\theta\right)$$

Notice a pattern? If we continue to do this iteratively, we get closer to the fixed point

$$V(k_t) = \frac{\theta}{1-\beta\theta}k_t^\theta + constant, \text{ and } k_{t+1} = \alpha(k_t) = \beta\theta k_t^\theta$$

Let's check the TVC:

$$\lim_{t\to\infty} \beta^t u'(c_t)k_{t+1} = \lim_{t\to\infty} \beta^t \frac{1}{c_t} k_{t+1} = \lim_{t\to\infty} \beta^t \frac{1}{k_t^\theta - \beta\theta k_t^\theta} \beta\theta k_t^\theta = \lim_{t\to\infty} \beta^t \frac{\beta\theta}{1-\beta\theta} = 0$$

Given k_0 and that the TVC holds, we can write the steady-state conditions:

$$k^* = k_{t+1} = \alpha(k_t) = \beta\theta k_t^\theta = \beta\theta k^{*\theta}$$
$$k^* = (\beta\theta)^{\frac{1}{1-\theta}}$$

5.2.2 Linear Technology

Let our agents be CRRA wiht $u(c) = \dfrac{c^{1-\alpha}-1}{1-\alpha}$ and $F(k) = Ak$. Our Bellman equation is:

$$V(k_t) = \max_{k_{t+1}} \; u(Ak_t - k_{t+1} + k_t(1-\delta)) + \beta V(k_{t+1})$$

Like the standard example, our $k_{t+1} = \alpha(k_t)$ should satisfy:

$$u'(c_t) = \beta u'(c_{t+1})[F'(k_{t+1}) + 1 - \delta] = \beta u'(c_{t+1}) \cdot \underbrace{[A + 1 - \delta]}_{\text{For simplicity, call this } B}$$

$$u'(c_t) = c_t^{-\alpha}$$
$$c_t = F(k_t) - k_{t+1} + (1-\delta)k_t = (A + 1 - \delta)k_t - k_{t+1}$$

So this Euler equation can be rewritten as:

$$\frac{1}{c_t^\alpha} = \frac{\beta B}{c_{t+1}^\alpha}$$
$$\Rightarrow \; \frac{1}{(Bk_t - k_{t+1})^\alpha} = \frac{\beta B}{(Bk_{t+1} - k_{t+2})^\alpha}$$

Since our technology is linear, we will assume that $k_{t+1} = \gamma k_t$. As the marginal product of capital is the same along k_t, it is reasonable that the economy maintains the capital stock at some proportional level depending on consumption needs.

But before we run wild with this assumption, let's first make sure that such decision rule satisfies the Euler equation:

$$\frac{1}{(Bk_t - \gamma k_t)^\alpha} = \frac{\beta B}{(B\gamma k_t - \gamma^2 k_t)^\alpha} \;\Rightarrow\; \gamma^\alpha = \beta B$$

So as long as $\gamma = (\beta B)^{\frac{1}{\alpha}}$, this assumption of the optimal decision rule will satisfy the Euler equation. We can rewrite the optimal decision rule as:

$$k_{t+1} = \alpha(k_t) = [\beta(A + 1 - \delta)]^{\frac{1}{\alpha}} k_t$$

From this, we can draw the **phase diagram**[20] with

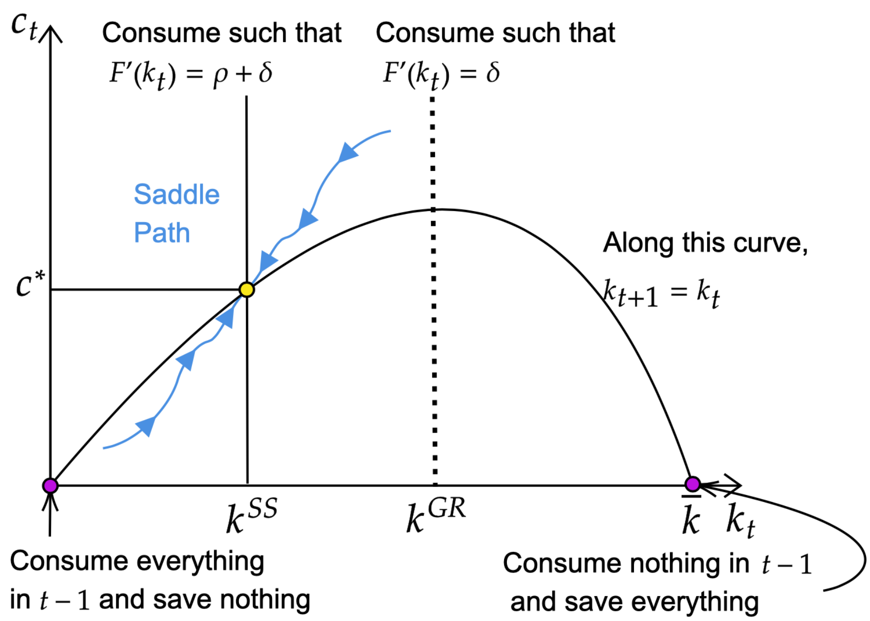
- Euler equation: $u'(c_t) = \beta u'(c_{t+1})[F'(k_{t+1}) + 1 - \delta]$
- Capital Accumulation: $k_{t+1} = F(k_t) - c_t + k_t(1 - \delta)$
  This can be written as $k_{t+1} - k_t = F(k_t) - c_t - \delta k_t$
- Assume that k_0 is known and that TVC holds

In the steady-state, we have

- $F'(k^*) = \rho + \delta$
- $c^* = F(k^*) - \delta k^*$

The black curve represents the equation $k_{t+1} = k_t$, which is equivalent to $c_t = F(k_t) - \delta k_t$.

The purple dots are 2 unstable steady-states, the yellow dot is a stable steady-state.



5.3 Mathematics Behind the Phase Diagram and Steady-State Equilibrium

5.3.1 Let's Get Used to Continuous Time

In the discrete time framework, our representative agent problem so far has been set up as:

$$\max_{c_t} \sum_{t=0}^{\infty} \beta^t u(c_t) \ s.t. \ c_t = F(k_t) - k_{t+1} + k_t(1 - \delta)$$

[20]We will revisit this phase diagram at a later time

Analogously, as the length of t gets smaller and smaller, we need to set up this present value discounted sums as an integral. Spoiler alert, this new present value evaluation is:

$$\max_c \int_0^\infty e^{-\rho t} u(c)\, dt \;\; s.t. \;\; \dot{k} = F(k) - c - \delta k$$

This may not seem intuitive, but we will go through the math right now to put away all doubts.

The first thing we are going to do is deal with this new discount factor $e^{-\rho t}$,

Rewriting $\boldsymbol{\beta}$: We want the discount factor in the two frameworks to be equivalent, so we should start with wanting one "length" of time to still discount exactly the same.

Let's break up one single time period into $n \in \mathbb{N}$ sub-periods of length Δ. We want to write a compound equivalent to β using discount factor[21] $\dfrac{\rho}{n}$:

$$\beta = \left(\frac{1}{1+\frac{\rho}{n}}\right)^n = \left(\frac{1}{1+\rho\Delta}\right)^{\frac{1}{\Delta}}$$

"Naturally", logging both sides of the equality, we get:

$$ln(\beta) = \frac{-ln(1+\rho\Delta)}{\Delta}$$

Now as $\Delta \to 0$, we approach continuous time, and can use L'Hospital's rule to find this limit:

$$ln(\beta) = \lim_{\Delta \to 0} \frac{-ln(1+\rho\Delta)}{\Delta} = \lim_{\Delta \to 0} \frac{\frac{d}{d\Delta} - ln(1+\rho\Delta)}{\frac{d}{d\Delta}\Delta} = \lim_{\Delta \to 0} \frac{-\frac{\rho}{1+\rho\Delta}}{1} = -\rho$$

Exponentiating both sides, we get that the continuous time counterpart of β is:

$$\beta = e^{ln(\beta)} = e^{-\rho}$$

As such, when we write the present value sum with β^t as the discount factor, we should feel right at home writing $e^{-\rho t}$ as the discount factor for the present value integral in continuous

[21] Note that this ρ is very different from the ρ of $\beta = \frac{1}{1+\rho}$ since the discount factor will be compounded an arbitrarily large amount of times.

times.

Rewriting resource constraints:
Obviously, this will be different from model to model, but readers should take solace in the fact the steps would be the same. Consider our classic constraint in a closed economy growth model:

$$c_t = F(k_t) - k_{t+1} + (1-\delta)k_t$$

which is equivalent to writing

$$\frac{k_{t+1} - k_t}{1} = F(k_t) - c_t - \delta k_t$$

If we simply think about the 1 on the LHS as 1 sub-period instead of 1 period, we get

$$\frac{k_{t+\Delta} - k_t}{\Delta} = F(k_t) - c_t - \delta k_t$$

Making Δ arbitrarily small (note that the δ here is not the same δ as in discrete time), we get

$$\dot{k} = F(k) - c - \delta k$$

The Social Planner's Problem in Continuous Time: Continuing our usage of n sub-periods with length Δ, our social planner's problem is

$$\max_c \sum_{t=0}^{\infty} e^{-\rho\Delta t} u(c)\ \Delta\ s.t.\ \frac{k_{t+\Delta} - k_t}{\Delta} = F(k_t) - c_t - \delta k_t$$

The state variable is k_t and the control variable is $k_{t+\Delta}$. We can set up the Bellman equation as:

$$V(k_t) = \max_{k_{t+\Delta}}\ u\left(F(k_t) - \frac{k_{t+\Delta} - k_t}{\Delta} - \delta k_t\right) + e^{-\rho\Delta} V(k_{t+\Delta})$$

The first order conditions (w.r.t. $k_{t+\Delta}$ and k_t) are

$$-\frac{1}{\Delta} u'(c_t) + e^{-\rho\Delta} V'(k_{t+\Delta}) = 0 \tag{1}$$

$$u'(c_t)[F'(k_t) + \frac{1}{\Delta} - \delta] = V'(k_t) \tag{2}$$

updating equation (2) and plug it into equation (1), we get

$$-\frac{1}{\Delta}u'(c_t) = -e^{-\rho\Delta}u'(c_{t+\Delta})[F'(k_{t+\Delta}) + \frac{1}{\Delta} - \delta]$$

Add $\frac{u'(c_{t+\Delta})}{\Delta}$ on both sides we get:

$$\frac{u'(c_{t+\Delta}) - u'(c_t)}{\Delta} = e^{-\rho\Delta}[F'(k_{t+\Delta}) - \delta]u'(c_{t+\Delta}) + \frac{e^{-\rho\Delta} - 1}{\Delta}u'(c_{t+\Delta})$$

Take the limit as $\Delta \to 0$ on both sides we get:

$$u''(c)\dot{c} = [F'(k) - \delta]u'(c) - \rho u'(c_{t+\Delta})$$

Dividing both sides by $u''(c)$ we get:

$$\dot{c} = \frac{u'(c)}{u''(c)}[F'(k) - \rho - \delta]$$

Similar to the discrete time case, we can pin down the solution if

(i) k_0 is given

(ii) **(TVC)** $\lim_{t\to\infty} e^{-\rho t}u'(c)k = 0$

5.3.2 Example with the Solow Model using C-D Produciton Function

Recall our set up in continuous time has the Cobb-Douglas production function

$$F(k) = Ak^\alpha$$

and the resource constraint

$$\dot{k} = sF(k) - \delta k = sAk^\alpha - \delta k$$

In steady-state, we must have $\dot{k} = 0$, meaning

$$sAk^\alpha = \delta k \quad \Rightarrow \quad k^{1-\alpha} = \frac{sA}{\delta} \quad \Rightarrow \quad k^* = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$$

Consider a different case where $y = Ak^{1-\alpha}$, then we can take the derivative of y w.r.t. t and get:

$$\dot{y} = A(1-\alpha)k^{-\alpha}\dot{k} = A(1-\alpha)k^{-\alpha}[sAk^{\alpha} - \delta k] = sA^2(1-\alpha) - A(1-\alpha)\delta k^{1-\alpha}$$
$$= sA^2(1-\alpha) - (1-\alpha)\delta y = (1-\alpha)[sA^2 - \delta y]$$

Dividing $sA^2 - \delta y$ and integrate w.r.t. t, we get:

$$\int_0^\infty \frac{dy/dt}{sA^2 - \delta y}\,dt = \int_0^\infty (1-\alpha)\,dt$$
$$\Rightarrow \int_0^\infty \frac{1}{sA^2 - \delta y}\,dy = (1-\alpha)t + C_1$$
$$\Rightarrow -\frac{1}{\delta}ln(sA^2 - \delta y) = (1-\alpha)t + C_1$$
$$\Rightarrow sA^2 - \delta y = e^{-\delta(1-\alpha)t}\cdot C_2$$
$$\Rightarrow y = \frac{1}{\delta}[sA^2 - C_2e^{-\delta(1-\alpha)t}],\ k = \left[\frac{1}{\delta}[sA^2 - C_2e^{-\delta(1-\alpha)t}]\right]^{\frac{1}{1-\alpha}}$$

5.3.3 Optimal Control Theory

Consider an integral constrained maximization problem (as opposed to the “discrete” ones we are used to using a Lagrangian for) set up as:

$$\max J(x,u) = \int_0^T F(x(t),u(t),t)\,dt\ s.t.\ \dot{x} = g(x,u,t)$$

where

- $x(t)$ is a vector of state variables
- $u(t)$ is a vector of control variables

Similar to how we “minimize” a Lagrangian, we can solve this problem with:

$$\max \int_0^T [F + \lambda(g - \dot{x})]dt$$

Notice that since we cannot choose the state we want to be in, having $\dot{x}$ in the equation really ties our hands. However, using integration by parts, we can write

$$-\int_0^T \lambda\dot{x}dt = -\lambda(t)x(t)\mid_0^T + \int_0^T x\dot{\lambda}dt = \int_0^T x\dot{\lambda}\,dt + \lambda(0)x(0) - \lambda(T)x(T)$$

So we can rewrite the problem as:

$$\max_u \int_0^T [F + \lambda g + x\dot{\lambda})]dt + \lambda(0)x(0) - \lambda(T)x(T) \tag{1}$$

Denote u^* as the **optimal control variable** and let $u(a) = (1-a)u^* + a \cdot u = u^* + ah$ denote any other control variable that deviates from the optimal one ($a \in [0,1]$, h is deviation).

Since u^* is the optimal control, it must be that $\left.\frac{\partial J}{\partial a}\right|_{a=0} = 0$ (otherwise, some other control could have been better). We can thus differentiate[22] equation (1) with respect to a:

$$\int \left[\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial a} + \frac{\partial F}{\partial u}\cdot h + \lambda\frac{\partial g}{\partial x}\cdot\frac{\partial x}{\partial a} + \lambda\frac{\partial g}{\partial u}\cdot h + \dot{\lambda}\frac{\partial x}{\partial a}\right] dt = 0$$

We can factor out the $\frac{\partial x}{\partial a}$ and h to get:

$$\int \frac{\partial x}{\partial a}\left[\frac{\partial F}{\partial x} + \lambda\frac{\partial g}{\partial x} + \dot{\lambda}\right] + h\left[\frac{\partial F}{\partial u} + \lambda\frac{\partial g}{\partial u}\right] dt = 0$$

One idea[23] to solve this complicated equation is to choose λ such that

$$\frac{\partial F}{\partial x} + \lambda\frac{\partial g}{\partial x} + \dot{\lambda} = 0$$

That way, we just need to solve for

$$\frac{\partial F}{\partial u} + \lambda\frac{\partial g}{\partial u} = 0$$

[22] In lecture, Andrei use $\frac{\partial y}{\partial a}$ (where y is a realization of x) instead of $\frac{\partial x}{\partial a}$. I personally find that makes it difficult to keep track since even if y is a realization, it sill is affected by the choice of controls. I typed this up in the way that I think is easier to understand, if you disagree, feel free to change those back to $\frac{\partial y}{\partial a}$.

[23] I asked Andrei in lecture about why it is intuitive to do this, and his answer is basically "You can do it other ways but this is easy".

This somewhat strange way of solving this kind of maximization is called a Present Value **Hamiltonian** ($\mathcal{H}$). Formally, we define the Hamiltonian to be the function you want to maximize plus the constraint times multiplier. Generally, it looks something like:

$$\mathcal{H} = F + \lambda \cdot g$$

More complicated than a Lagrangian, the optimal control variable must satisfy:

- $\frac{\partial \mathcal{H}}{\partial x} = -\dot{\lambda}$. The derivative of the Hamiltonian with respect to the state variable must equal the negative of the derivative of the multiplier with respect to time.
- $\frac{\partial \mathcal{H}}{\partial u} = 0$. The derivative of the Hamiltonian with respect to the control variable must equal 0.
- $\frac{\partial \mathcal{H}}{\partial \lambda} = g$. The derivative of the Hamiltonian with respect to the multipliers must equal the constraints.

The second and third conditions are basically the same as a Lagrangian. For the first condition, readers can think about it as an equivalent to the multipliers in the envelope theorem and $-\dot{\lambda}$ represents the change in F if there is a shock to the state variables.

Alternatively, we can write the Current Value **Hamiltonian** where $\mu = \lambda e^{\rho t}$:

$$\mathcal{H} = e^{-\rho t}[e^{\rho t}F + \mu g]$$

and the necessary conditions become:

- $\frac{\partial \mathcal{H}}{\partial x} = \rho\mu - \dot{\mu}$. The derivative of the Hamiltonian with respect to the state variable must equal the negative of the derivative of the multiplier with respect to time.
- $\frac{\partial \mathcal{H}}{\partial u} = 0$. The derivative of the Hamiltonian with respect to the control variable must equal 0.
- $\frac{\partial \mathcal{H}}{\partial \mu} = g$. The derivative of the Hamiltonian with respect to the multipliers must equal the constraints.

Example:

We want to solve the maximization problem:

$$\max_c \int_0^\infty e^{-\rho t} u(c)\, dt \ s.t. \ \dot{k} = F(k) - c - \delta k$$

In this problem, we have

$$F(x, u, t) = e^{-\rho t} u(c)$$
$$g(x, u, t) = F(k) - c - \delta k$$

So the Hamiltonian is:

$$\mathcal{H} = e^{-\rho t} u(c) + \lambda[F(k) - c - \delta k] = e^{-\rho t}[u(c) + \mu[F(k) - c - \delta k]]$$

Necessary Condition 1: $\dfrac{\partial \mathcal{H}}{\partial u} = 0$

$$\frac{\partial \mathcal{H}}{\partial c} = 0 \ \Rightarrow \ e^{-\rho t}[u'(c) - \mu] = 0 \ \Rightarrow \ \mu = u'(c)$$
$$\Rightarrow \ \dot{\mu} = u''(c)\dot{c} = \dot{\lambda} e^{\rho t} + \rho \lambda e^{\rho t} = e^{\rho t}[\dot{\lambda} + \rho\lambda]$$

Necessary Condition 2: $\dfrac{\partial \mathcal{H}}{\partial x} = -\dot{\lambda}$

$$\frac{\partial \mathcal{H}}{\partial k} = -\dot{\lambda} \ \Rightarrow \ \frac{\partial \mathcal{H}}{\partial k} = \lambda[F'(k) - \delta] = -\dot{\lambda}$$

Since $\dot{\mu} = u''(c)\dot{c} = e^{\rho t}[\dot{\lambda} + \rho\lambda] = e^{\rho t}[\rho\lambda - \lambda[F'(k) - \delta]]$, so

$$u''(c)\dot{c} = \mu[\rho - F'(k) + \delta] \ \Rightarrow \ \dot{c} = -\frac{u'(c)}{u''(c)}[F'(k) - \rho - \delta]$$

Necessary Condition 3: $\dfrac{\partial \mathcal{H}}{\partial \lambda} = g$

$$\frac{\partial \mathcal{H}}{\partial \lambda} = F(k) - c - \delta k = \dot{k}$$

5.3.4 Revisiting the Phase Diagram

Revising our phase diagram from earlier for continuous time. In steady-state, we have

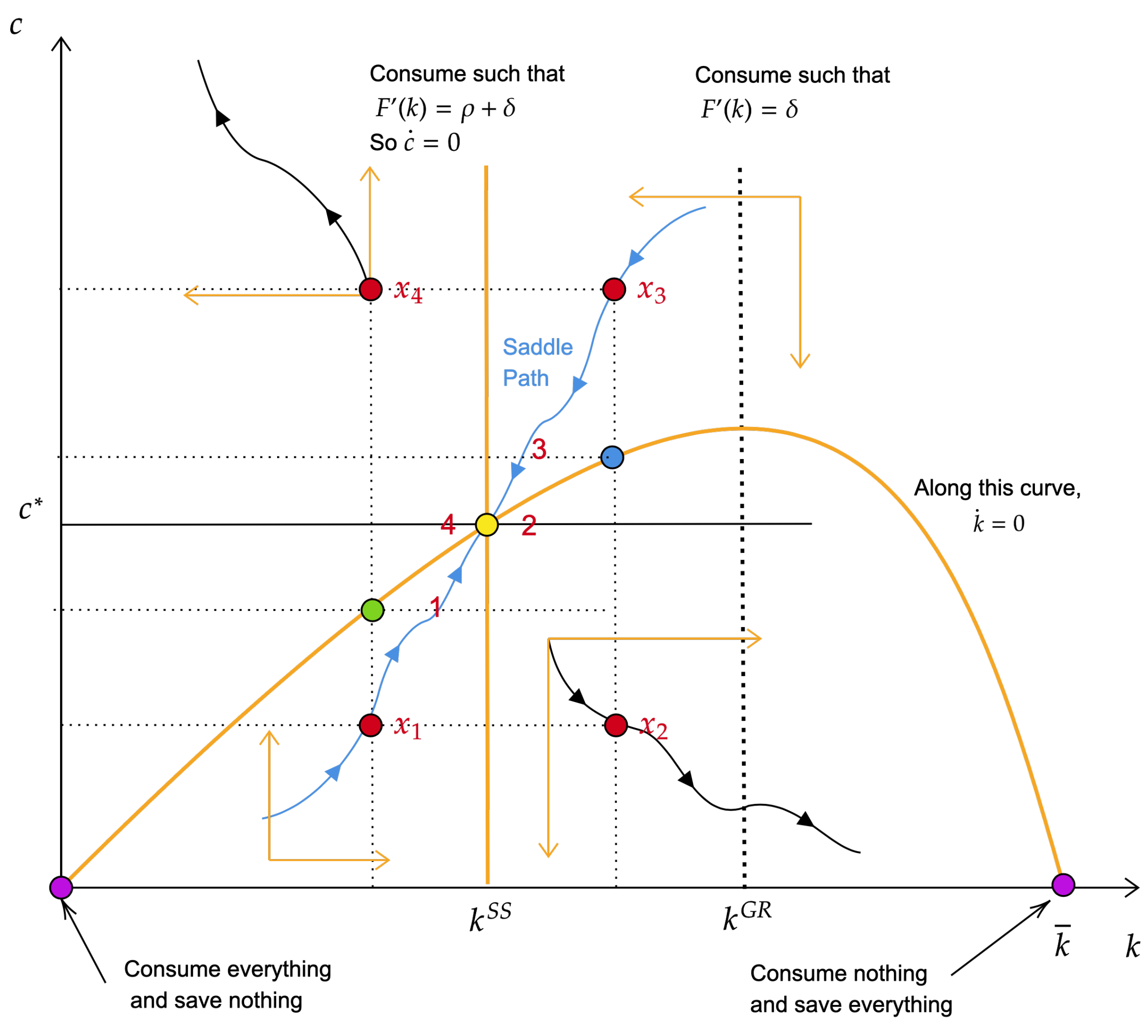
- $\dot{c} = 0 \;\Rightarrow\; F'(k^*) = \rho + \delta$
- $\dot{k} = 0$

To study this diagram, we need to first separate it into 4 quadrants, made by the curve $\dot{k} = 0$ and the line $F'(k) = \rho + \delta$.

- Above $\dot{k} = 0$ (x_3 and x_4 in the figure below), consumption is higher than what the capital can sustain (the blue and green points), so the capital would fall ($\dot{k} < 0$).
- Below $\dot{k} = 0$ (x_2 and x_1 in the figure below), consumption is lower than what the capital can sustain (the blue and green points), so the capital would grow ($\dot{k} > 0$).
- To the left of $F'(k) = \rho + \delta$, (x_1 and x_4 in the figure below), the marginal product is higher than what is needed to sustain consumption, so c would grow ($\dot{c} > 0$).
- To the right of $F'(k) = \rho + \delta$, (x_2 and x_3 in the figure below), the marginal product is lower than what is needed to sustain consumption, so c would fall ($\dot{c} < 0$).

Combining these, we know that

- For x_1 and any point in quadrant 1, $\dot{k} > 0$ and $\dot{c} > 0$
- For x_2 and any point in quadrant 2, $\dot{k} > 0$ and $\dot{c} < 0$
- For x_3 and any point in quadrant 3, $\dot{k} < 0$ and $\dot{c} < 0$
- For x_4 and any point in quadrant 4, $\dot{k} < 0$ and $\dot{c} > 0$



Recall our continuous time optimization conditions:

- $\dot{c} = -\frac{u'(c)}{u''(c)}[F'(k) - \rho - \delta$
- $\dot{k} = F(k) - c - \delta k$

The TVC ($\lim_{t\to\infty} e^{-\rho t}u'(c)k = 0$) and the assumption that $\lim_{k\to 0} MPK = \infty$, $\lim_{c\to 0} u'(c) = \infty$,. Given any $k \in (0, \bar{k})$, these conditions rule out things too far out of quadrants 2 and 4. So we know that the optimal growth path[24] must be something along the saddle path outlined in blue.

Our goal now is to try to use the Hamiltonians to find the saddle path. Recall the necessary conditions for the optimal control variable:

[24] Not the same as the BGP.

Present Value Hamiltonian	Current Value Hamiltonian
$\mathcal{H}^P = e^{-\rho t}u(c) + \lambda[F(k) - c - \delta k]$	$\mathcal{H}^C = e^{-\rho t}\{u(c) + \mu[F(k) - c - \delta k]\}$
$\frac{\partial \mathcal{H}^P}{\partial c} = 0$ $\frac{\partial \mathcal{H}^P}{\partial k} = -\dot{\lambda}$ $\frac{\partial \mathcal{H}^P}{\partial \lambda} = F(k) - c - \delta k$	$\frac{\partial \mathcal{H}^C}{\partial c} = 0$ $\frac{\partial \mathcal{H}^C}{\partial k} = \rho\mu - \dot{\mu}$ $\frac{\partial \mathcal{H}^C}{\partial \mu} = F(k) - c - \delta k$
$-\dot{\lambda} = \frac{\partial \mathcal{H}^P}{\partial k} = e^{-\rho t}\frac{\partial \mathcal{H}^C}{\partial k} = e^{-\rho t}[\rho\mu - \dot{\mu}]$	

Note that a lot of times, the necessary conditions are non-linear, so we would typically log-linearize the system and solve for the conditions.

Let us make our continuous time optimization conditions implicit functions and study how they behave:

- $\dot{c} = -\frac{u'(c)}{u''(c)}[F'(k) - \rho - \delta \equiv n(c, k)$
- $\dot{k} = F(k) - c - \delta k \equiv m(c, k)$

In steady-state, we must thus have $n(c^*, k^*) = m(c^*, k^*) = 0$. Using Taylor's theorem[25], we can locally approximate deviation from the steady-state.

Define $\Delta c = c - c^*$, $\Delta k = k - k^*$ as deviation from the steady-state. The first order approximation around (c^*, k^*) is

$$\dot{c} - c^* \approx \Delta\dot{c} = n(c^*, k^*) + n_c(c^*, k^*)\Delta c + n_k(c^*, k^*)\Delta k$$
$$\dot{k} - k^* \approx \Delta\dot{k} = m(c^*, k^*) + m_c(c^*, k^*)\Delta c + m_k(c^*, k^*)\Delta k$$

Now, at the steady-state, $n(c^*, k^*) = m(c^*, k^*) = 0$. So this system is really

$$\Delta\dot{c} \approx n_c(c^*, k^*)\Delta c + n_k(c^*, k^*)\Delta k$$
$$\Delta\dot{k} \approx m_c(c^*, k^*)\Delta c + m_k(c^*, k^*)\Delta k$$

[25] The first order approximation of $f(x)$ at $x = a$ is $f(x) \approx f(a) + f'(a)(x - a)$

We can write this as a linear system with matrices:

$$\begin{pmatrix} \Delta\dot{c} \\ \Delta\dot{k} \end{pmatrix} \approx \begin{pmatrix} n_c(c^*, k^*) & n_k(c^*, k^*) \\ m_c(c^*, k^*) & m_k(c^*, k^*) \end{pmatrix} \begin{pmatrix} \Delta c \\ \Delta k \end{pmatrix}$$

Notice that the middle 2×2 matrix is the Jacobin of the steady-state conditions. This system approximates the behavior of the growth around the saddle path around (c^*, k^*). To study this, we need to learn a little bit more about differential equations.

5.3.5 Solving a System of Differential Equations

Consider our standard transition equation $\dot{x} = ax = \frac{dx}{dt} \Rightarrow \int \frac{dx}{x} = \int a\,dt \Rightarrow ln(x) = at + C_1 \Rightarrow x = C_2 e^{at}$.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\begin{cases} x_1' = ax_1 + bx_2 \\ x_2' = cx_1 + dx_2 \end{cases} \Rightarrow X' = AX$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ How do we figure out what X is? We will guess and verify.

Let's try $X = ke^{rt}$, so[26] $\dot{X} = kre^{rt} = Ake^{rt} \Rightarrow kr = AR \Rightarrow (A - rI)k = 0$.
This means that the growth rate(s) r is the eigenvalue of the matrix A and k is the corresponding eigenvalue for each r.
For the solution (X) to exist, we need $det(A - rI) = 0$, giving us the characteristic equation $(a - r)(d - r) - bc = 0$. The solutions to this equation can be generalized into three cases:

(i) Two distinct real-valued solutions

(ii) Two complex conjugate solutions

(iii) Repeated solutions

We will focus our discussion on the first case.

Case 1: Two distinct real-valued solutions

Suppose we have the system $\begin{cases} x_1' = 3x_1 - 2x_2 \\ x_2' = 2x_1 - 2x_2 \end{cases}$ and initial condition $x(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

We can rewrite this as $X' = AX$ where $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$

[26] The equality to AXe^{rt} is from the general form of $X' = AX$, which is why we "guessed" X to be exponential

So the characteristic equation is $|A - rI| = (3-r)(-2-r)-(-4) = 0 \;\Rightarrow\; r_{1,2} = \{-1, 2\}$

(i) $r_1 = -1$

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} k_1^1 \\ k_2^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\Rightarrow\; 2k_1^1 - k_2^1 = 0 \;\Rightarrow\; \underbrace{k^1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\text{Eigenvector of } r_1 = -1}$$

(ii) $r_2 = 2$

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} k_1^1 \\ k_2^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\Rightarrow\; k_1^2 - 2k_2^2 = 0 \;\Rightarrow\; \underbrace{k^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\text{Eigenvector of } r_2 = 2}$$

And so the general solution to the system is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-1 \cdot t} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

From the initial condition, we can calculate C_1 and C_2 by solving:

$$X(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^0 + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^0 \;\Rightarrow\; C_1 = 1,\, C_2 = -1$$

Since all of these are derived from the eigenvalue of the coefficient matrix, it should be no surprise that we can qualify our solutions with conditions on the eigenvalues:

(i) Saddle Path: $r_1 < 0$, $r_2 > 0$ (Grows in 2 directions and falls in 2 directions)

(ii) Sink: $r_1, r_2 < 0$ (Falls in all 4 directions)

(iii) Source: $r_1, r_2 > 0$ (Grows in all 4 directions)

Case 2: Two complex conjugate solutions

Suppose our eigenvalues are $r = \lambda \pm \mu \cdot i$ with the eigenvectors $k = a \pm b \cdot i$, then we have (only doing the (+) eigenvector here):

$$ke^{rt} = (a + b_i)e^{\lambda t}e^{\mu i t} = (a + bi)e^{\lambda t} \underbrace{[cos(\mu t) + i \cdot sin(\mu t)]}_{e^{it} = cos(t) + i \cdot sin(t)}$$

$$= e^{\lambda y}[a \cdot cos(\mu t) + bi \cdot cos(\mu t) + ai \cdot sin(\mu t) - b \cdot sin(\mu t)]$$
$$= e^{\lambda t}[a \cdot cos(\mu t) - b \cdot sin(\mu t)] + i \cdot e^{\lambda t}[b \cdot cos(\mu t) + a \cdot sin(\mu t)]$$

Combined with the other eigenvector, we will get solutions that spiral around the initial condition (towards focus if real parts are negative.).

Case 3: Repeated solutions

Repeated solutions of the characteristic equation can mean either a complete system or a defective system. Think of repeated roots as lapping on the unit circle and you have to just guess how many laps it took. These are solvable but out of our scope.

Example: Class Example

Suppose we have the differential equation $y''' - 3y'' + 2y' - y = 0$, let $y = x_1$, $y' = x_2$, $y'' = x_3$, we can then rewrite this as: $x_3' = 3x_3 - 2x_2 + x_1$, giving us a system of 3 equations:

$$\begin{cases} x_1' = 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 \\ x_2' = 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 \\ x_3' = 1 \cdot x_1 - 2 \cdot x_2 + 3 \cdot x_3 \end{cases} \Rightarrow \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Using Wolfram Alpha, we can see that this system has 1 real positive eigenvalue and 2 complex conjugate eigenvalues. This means that the solution is an unstable spiral going away from the initial condition.

6 Endogenous Growth Theory

As we have seen in the last section, simply thinking about growth as $Y = AK^{\alpha}H^{1-\alpha}$ is not enough. What we have done is describe what the steady-state should look like, but we have yet to produce an explanation for how the economy gets on certain paths (e.g., the BGP). To do so, we must endogenize choices made by agents in the economy. There are 2 main thoughts on how that can be done:

1. Learning-or-Doing (Lucas): Endogenize the choices workers make about participation (H)
2. Research and Development (Romer): Endogenize the choices firms make on productivity (A)

To discuss the learning-or-doing model, we must first understand the context.

6.0 Background: Lucas AER (1990) *Why Doesn't Capital Flow from Rich to Poor Countries*

What inspired this framework is the realization that the marginal product of capital between different countries can be drastically different. In Lucas' observation, since India is less developed (has lower capital stock) than the US, it should be the case that $MPK^{India} > MPK^{US}$. But if that is the case, why does the force of economics not induce a flow of capital from US to India?
Let's try to explain this by making different assumptions about the production function.

Attempt 1: The production functions are the same US and India, but capital per capita k is what made the difference.

- Production Function: $Y = AK^\alpha N^{1-\alpha}$ with $\alpha \approx 0.4$
- From empirical studies, $\frac{y^{US}}{y^{India}} \approx 15$

Notice we have:

$$Y = AK^\alpha N^{1-\alpha} \;\Rightarrow\; y \equiv \frac{Y}{N} = A\left(\frac{K}{N}\right)^\alpha = Ak^\alpha \;\Rightarrow\; k = \left(\frac{y}{A}\right)^{\frac{1}{\alpha}}$$

So the MP"k" (using capital per capita) is

$$A\alpha k^{\alpha-1} = A\alpha\left(\frac{y}{A}\right)^{\frac{\alpha-1}{\alpha}} = A^{\frac{1}{\alpha}}\alpha y^{\frac{\alpha-1}{\alpha}}$$

and so the ratio of MPk between US and India is

$$\frac{MPK^{India}}{MPK^{US}} = \left(\frac{y^{US}}{y^{India}}\right)^{\frac{1-\alpha}{\alpha}} = 15^{\frac{0.6}{0.4}} \approx 58$$

Using this logic, the ratio of the MPk between US and India is huge, even though we don't see capital flow from US to India.

What if we have been measuring capital incorrectly?

Attempt 2: What if we scale capital by how effective the workers are?

- Production Function: $Y = AK^\alpha (hN)^{1-\alpha}$ with $\alpha \approx 0.4$
- From empirical studies, $\frac{h^{US}}{h^{India}} \approx 5$ where $H = h \cdot N$ is the effective count of labor

Notice we have:

$$Y = AK^\alpha (hN)^{1-\alpha} \Rightarrow y \equiv \frac{Y}{N} = A\left(\frac{K}{N}\right)^\alpha h^{1-\alpha} = A\left(\frac{k}{h}\right)^\alpha h \Rightarrow \left(\frac{k}{h}\right) = \left(\frac{y}{Ah}\right)^{\frac{1}{\alpha}}$$

So the MPk (using capital per capita) is

$$A\alpha k^{\alpha-1} h^{1-\alpha} = A\alpha \left(\frac{k}{h}\right)^{\alpha-1} = A\alpha \left(\frac{y}{Ah}\right)^{\frac{\alpha-1}{\alpha}} = A^{\frac{1}{\alpha}}\alpha \left(\frac{y}{h}\right)^{\frac{\alpha-1}{\alpha}}$$

and so the ratio of MPk between US and India is

$$\frac{MPK^{India}}{MPK^{US}} = \left(\frac{y^{US}}{y^{India}} \cdot \frac{h^{India}}{h^{US}}\right)^{\frac{1-\alpha}{\alpha}} = (15 \cdot \frac{1}{5})^{\frac{0.6}{0.4}} \approx 5$$

Using this logic, the ratio of the MPk between US and India is smaller but still at the factor of 5, even though we don't see capital flow from US to India.

Attempt 3: What if there are externalities in having an optimal average human capital?

- Production Function: $Y = AK^\alpha (hN)^{1-\alpha} h_{av}^\gamma$ where h_{av} is the average h in the economy
- Let $y \equiv \frac{Y}{hN}$ and $k \equiv \frac{K}{hN}$

Notice we have:

$$Y = AK^\alpha (hN)^{1-\alpha} h_{av}^\gamma \Rightarrow y \equiv \frac{Y}{hN} = A\left(\frac{K}{hN}\right)^\alpha h_{av}^\gamma = Ak^\alpha h_{av}^\gamma \Rightarrow k = \left(\frac{y}{Ah_{av}^\gamma}\right)^{\frac{1}{\alpha}}$$

So the MPk (using capital per capita) is

$$A\alpha k^{\alpha-1} h_{av}^\gamma = A\alpha \left(\frac{y}{Ah_{av}^\gamma}\right)^{\frac{\alpha-1}{\alpha}} h_{av}^\gamma = A^{\frac{1}{\alpha}} \alpha y^{\frac{\alpha-1}{\alpha}} h_{av}^{\frac{\gamma}{\alpha}}$$

and so the ratio of MPk between US and India is

$$\frac{MPK^{India}}{MPK^{US}} = \underbrace{\left(\frac{y^{US}}{y^{India}}\right)^{\frac{1-\alpha}{\alpha}}}_{\approx 5 \text{ from attempt 2}} \cdot \left(\frac{h_{av}^{India}}{h_{av}^{US}}\right)^{\frac{\gamma}{\alpha}} \approx 5 \cdot \left(\frac{1}{5}\right)^{\frac{\gamma}{0.4}}$$

The only thing left to solve now is γ. We will do this by first log-linearizing the production function:

$$ln(Y) = ln(A) + \alpha ln(K) + (1-\alpha)ln(h) + (1-\alpha)ln(N) + \gamma ln(h_{av})$$

In equilibrium, an economy's h should be equal to it's h_{av}, so we can rewrite this as

$$ln(Y) = ln(A) + \alpha ln(K) + (1-\alpha+\gamma)ln(h) + (1-\alpha)ln(N)$$

Differentiating this with respect to time we get:

$$\frac{\dot{Y}}{Y} = \alpha\frac{\dot{K}}{K} + (1-\alpha+\gamma)\frac{\dot{h}}{h} + (1-\alpha)\frac{\dot{N}}{N}$$
$$g_Y = \alpha g_K + (1-\alpha+\gamma)g_h + (1-\alpha)g_N$$

Empirically, $g_h = \frac{\dot{h}}{h} \approx 0.09$ and $\gamma \approx 0.38$, and so the ratio of MPk between US and India is

$$\frac{MPK^{India}}{MPK^{US}} = \underbrace{\left(\frac{y^{US}}{y^{India}}\right)^{\frac{1-\alpha}{\alpha}}}_{\approx 5 \text{ from attempt 2}} \cdot \left(\frac{h_{av}^{India}}{h_{av}^{US}}\right)^{\frac{\gamma}{\alpha}} \approx 5 \cdot \left(\frac{1}{5}\right)^{\frac{0.38}{0.4}} \approx 1$$

Seems like we found something that will stick. Let's try it out in the model!

6.1 Learning or Doing (Lucas, JME(1988), *On the Mechanics of Economics Development*)

Remember, our goal is to explain, possibly through externality of average human capital, why capital doesn't flow from rich countries to poor countries. In other words, why is it that all economies don't converge to k^*.

We will begin by setting up the **Environment** of the model:

- N consumers with utility function $u(C) = \dfrac{C^{1-\sigma}-1}{1-\sigma}$

- Production function: $Y = AK^{\alpha}(uhN)^{1-\alpha}h_{av}^{\gamma}$
- Constraint 1: $\dot{K} = Y - NC$ (no depreciation)
- Constraint 2: $\dfrac{\dot{h}}{h} = \delta(1-u)$ where u denotes the fraction of time spent working, $1-u$ denotes the fraction of time spent improving human capital, and δ is just some constant

6.1.1 The Social Planner's Problem

The Social Planner's Problem is:

$$\max \int_0^{\infty} e^{-\rho t} N \frac{C^{1-\sigma}-1}{1-\sigma}\, dt$$

$$s.t. \begin{cases} \dot{K} = Y - NC \\ \frac{\dot{h}}{h} = \delta(1-u) \end{cases}$$

In this system, we have 2 state variables (K, h) and 2 control variables (C, u), giving us 4 equations to describe the equilibrium path.

Using Current Value Hamiltonian, we can set up

$$\mathcal{H}^C = e^{-\rho t}\left\{ N\frac{C^{1-\sigma}-1}{1-\sigma} + \theta_1[AK^{\alpha}(uN)^{1-\alpha}h^{1-\alpha+\gamma} - NC] + \theta_2[h\delta(1-u)] \right\}$$

Notice that the social planner wants the equilibrium level of human capital, so $h = h_{av}$ in the production function.

Since we have 2 state and 2 control variables, we need 4 necessary conditions (barring the constraint conditions):

- $\dfrac{\partial \mathcal{H}^C}{\partial C} = 0 = NC^{-\sigma} - \theta_1 N$
- $\dfrac{\partial \mathcal{H}^C}{\partial u} = 0 = \theta_1[AK^{\alpha}(1-\alpha)u^{-\alpha}N^{1-\alpha}h^{1-\alpha+\gamma}] + \theta_2[-h\delta]$
- $\dfrac{\partial \mathcal{H}^C}{\partial K} = \rho\theta_1 - \dot{\theta}_1 = \theta_1[A\alpha K^{\alpha-1}(uN)^{1-\alpha}h^{1-\alpha+\gamma} - NC]$
- $\dfrac{\partial \mathcal{H}^C}{\partial h} = \rho\theta_2 - \dot{\theta}_2 = \theta_1[AK^{\alpha}(uN)^{1-\alpha}(1-\alpha+\gamma)h^{\gamma-\alpha}] + \theta_2[\delta(1-u)]$

Now, recall that on the BGP, all variables grow at a constant rate, so $g_h = \frac{\dot{h}}{h} = \delta(1-u)$ is a constant (we will call this constant ν). This means that u is also a constant on the BGP. Let's solve this system!

Condition (i): $\dfrac{\partial \mathcal{H}^C}{\partial C} = 0 = NC^{-\sigma} - \theta_1 N$

From the condition, we have $\theta_1 = C^{-\sigma}$, which is the marginal utility. Differentiate this w.r.t. time, we get

$$\dot{\theta}_1 = -\sigma C^{-1-\sigma}\dot{C} = -\sigma C^{-\sigma}\frac{\dot{C}}{C} = -\sigma \underbrace{C^{-\sigma}}_{=\theta_1} g_C$$

$$\Rightarrow \; g_{\theta_1} = \frac{\dot{\theta}_1}{\theta_1} = -\sigma \underbrace{g_C}_{\text{Call this } \kappa}$$

Condition (ii): $\dfrac{\partial \mathcal{H}^C}{\partial K} = \rho\theta_1 - \dot{\theta}_1 = \theta_1[A\alpha K^{\alpha-1}(uN)^{1-\alpha}h^{1-\alpha+\gamma} - NC]$

From the condition, we have $\rho\theta_1 - \dot{\theta}_1 = \theta_1 \cdot MPK$, so $\rho - g_{\theta_1} = MPK$.

So on the BGP, $MPK = \rho - g_{\theta_1} = \rho + \sigma\kappa$ is constant. Let's log-linearize this:

$$ln(A) + ln(\alpha) + (\alpha-1)ln(K) + (1-\alpha)ln(uN) + (1-\alpha+\gamma)ln(h) - ln(NC) = ln(\rho+\sigma\kappa)$$

Differentiate this w.r.t. time, we get:

$$\begin{aligned}
\frac{\alpha-1}{K}\dot{K} + \frac{1-\alpha}{N}\dot{N} + \frac{(1-\alpha+\gamma)}{h}\dot{h} &= 0 \\
(\alpha-1)g_K + (1-\alpha)g_N + (1-\alpha+\gamma)g_h &= 0 \\
(\alpha-1)g_K + (1-\alpha)\lambda + (1-\alpha+\gamma)\nu &= 0
\end{aligned}$$

where $\lambda = \dfrac{\dot{N}}{N}$ and $\nu = \dfrac{\dot{h}}{h} = \delta(1-u)$

The social planner is faced with the constraint

$$\begin{aligned}
\dot{K} = Y - NC &= AK^{\alpha}(uN)^{1-\alpha}h^{1-\alpha+\gamma} - NC \\
&= \frac{K}{\alpha}A\alpha K^{\alpha-1}(uN)^{1-\alpha}h^{1-\alpha+\gamma} - NC
\end{aligned}$$

$$= \frac{K \cdot MPK}{\alpha} - NC$$

$$g_K = \frac{\dot{K}}{K} = \frac{MPK}{\alpha} - \frac{NC}{K}$$

On the BGP, since MPK and g_K are both constant, it must be that $\frac{NC}{K}$ is also constant and so $ln\left(\frac{NC}{K}\right) = ln(N) + ln(C) - ln(K)$ is constant.

Differentiating w.r.t. time, we get $\frac{\dot{N}}{N} + \frac{\dot{C}}{C} - \frac{\dot{K}}{K} = \lambda + \kappa - g_K = 0$ so $g_k = \lambda + \kappa$

Condition (iii): $\frac{\partial \mathcal{H}^C}{\partial u} = 0 = \theta_1[AK^\alpha(1-\alpha)u^{-\alpha}N^{1-\alpha}h^{1-\alpha+\gamma}] + \theta_2[-h\delta]$

From the condition, we have

$$\frac{\theta_1}{\theta_2} = \frac{h\delta}{AK^\alpha(1-\alpha)u^{-\alpha}N^{1-\alpha}h^{1-\alpha+\gamma}} \tag{1}$$

Log-linearize this equation and differentiate w.r.t. time, we get:

$$\begin{aligned} g_{\theta_1} - g_{\theta_2} &= g_h - \alpha g_K - (1-\alpha)g_N - (1-\alpha+\gamma)g_h \\ g_{\theta_2} + \nu &= g_{\theta_1} + \alpha g_K + (1-\alpha)\lambda + (1-\alpha+\gamma)\nu \end{aligned} \tag{2}$$

Condition (iv): $\frac{\partial \mathcal{H}^C}{\partial c} = \rho\theta_2 - \dot{\theta}_2 = \theta_1[AK^\alpha(uN)^{1-\alpha}(1-\alpha+\gamma)h^{\gamma-\alpha}] + \theta_2[\delta(1-u)]$

From this condition, we have:

$$\rho - g_{\theta_2} = \frac{\theta_1}{\theta_2}[AK^\alpha(uN)^{1-\alpha}(1-\alpha+\gamma)h^{\gamma-\alpha}] + \delta(1-u)$$

Subbing in equation (1), we get

$$\begin{aligned} \rho - g_{\theta_2} &= \frac{h\delta \cdot [AK^\alpha(uN)^{1-\alpha}(1-\alpha+\gamma)h^{\gamma-\alpha}]}{AK^\alpha(1-\alpha)u^{-\alpha}N^{1-\alpha}h^{1-\alpha+\gamma}} + \delta(1-u) \\ &= \frac{\delta u(1-\alpha+\gamma)}{(1-\alpha)} + \delta(1-u) \\ &= \frac{\delta u(1-\alpha+\gamma)}{(1-\alpha)} + \nu \end{aligned}$$

Rewriting this we get:

$$g_{\theta_2} + \nu = \rho - \frac{\delta u(1-\alpha+\gamma)}{(1-\alpha)} = \underbrace{g_{\theta_1} + \alpha g_K + (1-\alpha)\lambda + (1-\alpha+\gamma)\nu}_{\text{From equation (2)}}$$

And we know g_{θ_1} from condition (i), so

$$\rho - \frac{\delta u(1-\alpha+\gamma)}{(1-\alpha)} = -\sigma\kappa + \alpha g_K + (1-\alpha)\lambda + (1-\alpha+\gamma)\nu \tag{3}$$

From condition (ii), we know $g_k = \lambda + \kappa$ and $(\alpha - 1)g_K + (1-\alpha)\lambda + (1-\alpha+\gamma)\nu = 0$, so we can simplify equation (3) to get:

$$\rho - \frac{\delta u(1-\alpha+\gamma)}{(1-\alpha)} = -\sigma\kappa + g_K + \underbrace{(\alpha-1)g_K + (1-\alpha)\lambda + (1-\alpha+\gamma)\nu}_{=0}$$

$$\Rightarrow \; \rho + (\sigma-1)\kappa - \lambda = \frac{\delta u(1-\alpha+\gamma)}{1-\alpha} \tag{4}$$

Notice that condition (ii) also implies

$$\begin{aligned} -(\alpha-1)g_K &= (1-\alpha)\lambda + (1-\alpha+\gamma)\nu \\ \Rightarrow \; g_K &= \lambda + \frac{(1-\alpha+\gamma)\nu}{1-\alpha} = \lambda + \kappa \\ \Rightarrow \; \kappa &= \frac{(1-\alpha+\gamma)\nu}{1-\alpha} \end{aligned}$$

Putting this into equation (4), we get:

$$\begin{aligned} \rho - \lambda + \frac{(\sigma-1)(1-\alpha+\gamma)}{1-\alpha}\nu &= \frac{\delta u(1-\alpha+\gamma)}{1-\alpha} \\ \nu &= \frac{\delta u}{\sigma-1} - \frac{1-\alpha}{(\sigma-1)(1-\alpha+\gamma)}(\rho-\lambda) \end{aligned}$$

As such, we can write

$$\begin{aligned} \nu &= -\frac{\delta(1-u)}{\sigma-1} + \frac{\delta}{\sigma-1} - \frac{1-\alpha}{(\sigma-1)(1-\alpha+\gamma)}(\rho-\lambda) \\ &= -\frac{\nu}{\sigma-1} + \frac{\delta}{\sigma-1} - \frac{1-\alpha}{(\sigma-1)(1-\alpha+\gamma)}(\rho-\lambda) \\ \frac{\sigma-1+1}{\sigma-1}\nu &= \frac{\delta}{\sigma-1} - \frac{1-\alpha}{(\sigma-1)(1-\alpha+\gamma)}(\rho-\lambda) \end{aligned}$$

$$\nu^{SP} = \frac{1}{\sigma}\left[\delta - \frac{1-\alpha}{1-\alpha+\gamma}(\rho - \lambda)\right]$$

Once we pinned down ν^{SP}, we also pinned down $\kappa^{SP} = \frac{1-\alpha+\gamma}{1-\alpha}\nu^{SP}$. So the social planner equilibrium is fully characterized by:

$$\begin{cases} \nu^{SP} &= \frac{1}{\sigma}\left[\delta - \frac{1-\alpha}{1-\alpha+\gamma}(\rho-\lambda)\right] \\ \kappa^{SP} &= \frac{1}{\sigma}\left[\frac{1-\alpha+\gamma}{1-\alpha}\delta - \rho + \lambda\right] \end{cases}$$

where ν^{SP} describes the optimal policy for the growth of human capital, and κ^{SP} describes the optimal policy for the growth of consumption.

6.1.2 The General Equilibrium Problem (Solved thanks to Sang Joon Rhee)

Now that we have solved the social planner version, let's try to solve the competitive equilibrium version.

Representative Households[27]:

Recall that our representative **household** problem is (Assuming that agents own the capital):

$$\max \int_0^\infty e^{-\rho t} \cdot \boldsymbol{N} \cdot \frac{c^{1-\sigma} - 1}{1-\sigma}\, dt$$

$$s.t. \begin{cases} \dot{k} = \boldsymbol{(r-\lambda)}k + w \cdot uh - c \\ \frac{\dot{h}}{h} = \delta(1-u) \end{cases}$$

Since this is an agent problem, the variables are written in per capita form. We have 2 state variables ($\{k, h\}$) and 2 control variables ($\{c, u\}$). The current value Hamiltonian of this

[27] In the Lucas paper, we need to solve the problem for all N households since the population can be growing. Also note that the capital accumulation equation is hence different. We have (since $(\dot{Nk}) = \dot{N}k + N\dot{k}$:

$$\dot{K} = rK + wuhN - Nc$$
$$\Rightarrow \ (\dot{Nk}) = rK + wuhN - Nc$$
$$\Rightarrow \ N\dot{k} = rNk + wuhN - Nc - \dot{N}k$$
$$\Rightarrow \ \dot{k} = rk + wuh - c - \lambda k = (r-\lambda)k + wuh - c$$

problem is:

$$\mathcal{H}^C = e^{-\rho t}\left\{\frac{c^{1-\sigma}-1}{1-\sigma} + \theta_1[rk+uwh-c)] + \theta_2[h\delta(1-u)]\right\}$$

This gives us 4 necessary conditions for the optimal policy for consumption and human capital accumulation:

(i) $\dfrac{\partial \mathcal{H}^C}{\partial c} = 0 = Nc^{-\sigma} - \theta_1$

(ii) $\dfrac{\partial \mathcal{H}^C}{\partial k} = \rho\theta_1 - \dot{\theta}_1 = \theta_1[r-\lambda]$

(iii) $\dfrac{\partial \mathcal{H}^C}{\partial u} = 0 = \theta_1[wh] + \theta_2[-h\delta]$

(iv) $\dfrac{\partial \mathcal{H}^C}{\partial h} = \rho\theta_2 - \dot{\theta}_2 = \theta_1[uw] + \theta_2[\delta(1-u)]$

Seems much simpler! Let's solve this like the social planner case:

Condition (i): $\dfrac{\partial \mathcal{H}^C}{\partial c} = 0 = Nc^{-\sigma} - \theta_1$

Directly, we get $Nc^{-\sigma} = \theta_1$ Differentiate w.r.t. time we get $\dot{\theta}_1 = -\sigma c^{-1-\sigma}\dot{c}N + \dot{N}c^{-\sigma} = -\sigma\theta_1\kappa + \dot{N}\frac{\theta_1}{N}$ Divide θ_1 on both sides we get $g_{\theta_1} = -\sigma\kappa + \lambda$

Condition (ii): $\dfrac{\partial \mathcal{H}^C}{\partial k} = \rho\theta_1 - \dot{\theta}_1 = \theta_1[r-\lambda]$

Dividing θ_1 on both sides, we get:

$$\rho + \underbrace{\sigma\kappa}_{=-g_{\theta_1}} - \lambda = r - \lambda \;\Rightarrow\; \kappa = \frac{r-\rho}{\sigma} \tag{1}$$

Condition (iii): $\dfrac{\partial \mathcal{H}^C}{\partial u} = 0 = \theta_1[wh] + \theta_2[-h\delta]$

Log-linearize this equation (moving $\theta_2[-h\delta]$ to LHS first) and then differentiate w.r.t. time we get:

$$g_{\theta_1} + g_h + g_w = g_{\theta_2} + g_h \;\Rightarrow\; g_{\theta_1} + g_w = g_{\theta_2} \tag{2}$$

Also from this condition, we get

$$\frac{\theta_1}{\theta_2} = \frac{\delta}{w} \tag{3}$$

Condition (iv): $\dfrac{\partial \mathcal{H}^C}{\partial h} = \rho\theta_2 - \dot{\theta_2} = \theta_1[uw] + \theta_2[\delta(1-u)]$

Dividing θ_2 on both sides, we get:

$$\begin{aligned} \rho - g_{\theta_2} &= \frac{\theta_1}{\theta_2} uw + \delta(1-u) \\ g_{\theta_2} &= \rho - \delta u - \delta(1-u) = \rho - \delta \end{aligned} \tag{4}$$

Subbing in equation (4) into euation (2) we get:

$$g_w = g_{\theta_2} - g_{\theta_1} = \rho - \delta + \sigma\kappa - \lambda \tag{5}$$

$$= \rho - \delta + r - \rho - \lambda = r - \delta - \lambda \tag{6}$$

So now are goal is to solve the representative firm's problem to find $\nu = g_h$.

Firms: The firms' problem in this economy is:

$$\max \int_0^\infty A(Nk)^\alpha (uhN)^{1-\alpha} h_{av}^\gamma - rNk - wuhN \, dt$$

In this competitive market, firms decides, in each period, how many workers to employ (N) and how much capital to rent (K).

The F.O.C.s of this problem is thus:

$$\alpha A(K)^{\alpha-1}(uhN)^{1-\alpha} h_{av}^\gamma = r$$

$$(1-\alpha)A(K)^\alpha (uhN)^{-\alpha} h_{av}^\gamma = w$$

Log-linearizing and differentiating both conditions w.r.t. time, we get:

$$g_r = (\alpha - 1)g_K + (1-\alpha)(\nu + \lambda) + \gamma g_h$$

$$g_w = \alpha g_K - \alpha(\nu + \lambda) + \gamma g_h$$

From equation (6), we know that, on the BGP, r is a constant, meaning that $g_r = 0$, so

$$g_K = (1 + \frac{\gamma}{1-\alpha})\nu + \lambda \tag{7}$$

$$\rho - \delta + \sigma\kappa - \lambda = \alpha g_K + (\gamma - \alpha)\nu - \alpha\lambda \tag{8}$$

From the capital accumulation formula (subbing in r and w from the firms' constraint), we have

$$\dot{k} = (r - \lambda)k + wuh - c = \underbrace{k \cdot MPK}_{=\alpha \frac{Y}{N}} - k\lambda + \underbrace{MPN \cdot uh}_{=(1-\alpha)\frac{Y}{N}} - c$$

Recall from earlier that $\dot{K} = (\dot{Nk}) = \dot{N}k + \dot{k}N \;\Rightarrow\; \dot{k} = k \cdot g_K - k\lambda$. So we have

$$g_K = \frac{Y}{Nk} - \frac{c}{k}$$

Notice that $\frac{Y}{Nk} = \frac{Y}{K}$ is just $\frac{1}{\alpha}MPK$, so on the BGP, $\frac{c}{k}$ is constant, so $g_k = g_c = \kappa$, meaning that $g_K = g_k + \lambda = \kappa + \lambda$. Subbing this into equation (8), we get:

$$\rho - \delta + \sigma\kappa - \lambda = \alpha\kappa + \alpha\lambda + (\gamma - \alpha)\nu - \alpha\lambda \;\Rightarrow\; \kappa = \frac{1}{\sigma - \alpha}[(\gamma - \alpha)\nu - \rho + \delta + \lambda]$$

Putting this in equation (7), we get

$$\begin{aligned} g_K - \lambda = \kappa &= \frac{1}{\sigma - \alpha}[(\gamma - \alpha)\nu - \rho + \delta + \lambda] = \frac{1 - \alpha + \gamma}{1 - \alpha}\nu \\ \Rightarrow\; \nu^{CE} &= \frac{1 - \alpha}{\sigma - \alpha + (\sigma - 1)(\gamma - \alpha)}[\lambda + \delta - \rho] \\ \Rightarrow\; \kappa^{CE} &= \frac{1 - \alpha + \gamma}{\sigma(1 - \gamma) - \gamma(1 - \alpha)}[\lambda + \delta - \rho] \end{aligned}$$

So when $\sigma = 1$, we have $\nu^{CE} = \delta - (\rho - \lambda) < \delta - \frac{1-\alpha}{1-\alpha+\gamma}(\rho - \lambda) = \nu^{SP}$. This makes sense since the representative households don't benefit from the human capital externality, they would invest less in human capital than what the social planner wanted.

6.2 Research and Development (Romer, JPE(1990), *Endogenous Technological Change*

The idea of this paper is simple. What if the economy endogenously decide what the total factor productivity is based on the states of the economy?
Suppose that's true, let's set up the **Environment:**

- There are 3 sectors in the economy:
  (1) Final goods sector (Competitive, population H_Y)
  (2) Intermediate Goods Sector (Imperfect competition, this is needed as firms in perfectly competitive markets make 0 profits and hence would not be able to invest in research)
  (3) Research sector: produce ideas (competitive labor market, population H_A)
- Total factor productivity is represented by the number of "ideas" A and it follows $\dot{A} = A\delta H_A$
- Production of intermediate goods uses a one-to-one technology using capital
- Production final goods follows $Y = H_Y^{1-\alpha} \int_0^A x_i^\alpha d_i$
- On BGP $g_A = g_K = g_Y$
- Assume constant population

6.2.1 The General Equilibrium Approach

Final Goods Sector Equilibrium:
The profit maximization problem in the final goods sector is:

$$\max_{H_Y, x_i} H_Y^{1-\alpha} \int_0^A x_i^\alpha \, d_i - w_Y H_Y - \int_0^A p_i x_i \, d_i$$

The F.O.C.s are:

$$(1-\alpha) H_Y^{-\alpha} \int_0^A x_i^\alpha \, d_i = w_Y = \frac{(1-\alpha)Y}{H_Y} \tag{1}$$

$$H_Y^{1-\alpha} \alpha x_i^{\alpha-1} = p_i \tag{2}$$

Intermediate Goods Sector Equilibrium:
The maximization problem is:

$$\max_{x_i} [p_i(x_i) - r] \cdot x_i$$

The first order condition is:

$$p_i'(x_i)x_i + p_i(x_i) = r \;\Rightarrow\; \frac{p_i'(x_i)x_i}{p_i(x_i)} + 1 = \frac{r}{p_i(x_i)}$$

From the final goods sector equation (2)[28], we know that

$$p(x_i) = H_Y^{1-\alpha}\alpha x_i^{\alpha_1} \;\Rightarrow\; \frac{p_i'(x_i)x_i}{p_i(x_i)} = \frac{H_Y^{1-\alpha}\alpha(\alpha-1)x_i^{\alpha-2}x_i}{H_Y^{1-\alpha}\alpha x_i^{\alpha-1}} = \alpha - 1$$

So we can rewrite the F.O.C. here as:

$$\alpha - 1 + 1 = \frac{r}{p_i(x_i)} \;\Rightarrow\; p_i(x_i) = \frac{r}{\alpha}$$

So the prices of all intermediate goods are the same. This means that in the intermediate goods market, we have:

- $K = \int_0^A x_i \, di = xA \;\Rightarrow\; x = \frac{K}{A}$
- $\pi = (p-r)x = \frac{1-\alpha}{\alpha} rx = \frac{1-\alpha}{\alpha} r \frac{K}{A}$

We can obtain r through calculating production:

$$Y = H_Y^{1-\alpha}\int_0^A x_i^\alpha \, di \;\Rightarrow\; Y = H_Y^{1-\alpha}\int_0^A x^\alpha \, di = H_Y^{1-\alpha}Ax^\alpha = H_Y^{1-\alpha}A\left(\frac{K}{A}\right)^{1-\alpha} K^\alpha = (AH_Y)^{1-\alpha}K^\alpha$$

Voila! This turns out to be a Cobb-Douglas Production Function!
We can thus rewrite equation (2):

$$\underbrace{(AH_Y)^{1-\alpha}\alpha K^{\alpha-1}}_{\frac{\partial}{\partial x}AH_Y^{1-\alpha}\left(\frac{K}{A}\right)^\alpha} = p = \frac{r}{\alpha} \;\Rightarrow\; \frac{\alpha Y}{K} = \frac{r}{\alpha} \;\Rightarrow\; r = \frac{\alpha^2 Y}{K}$$

So the profit in the intermediate goods sector is:

$$\pi = \frac{1-\alpha}{\alpha}\frac{\alpha^2 Y}{K}\frac{K}{A} = \alpha(1-\alpha)\frac{Y}{A} \tag{3}$$

As such, on the BGP, the profit is constant ($g_Y = g_A$)

[28] $p = \frac{r}{\alpha} > r \;\Rightarrow\; \pi = (p-r)x_i > 0 \;\Rightarrow\;$ firms can invest on research.

Research Sector Equilibrium:
$\dot{A} = \delta A H_A$ means each researcher develops δA ideals in each instant of time. Assume that the labor market is competitive, then it must be that $w_A = \delta A p_A$ where p_A is the price of ideas and w_A is the wage.

Since the market is competitive, it must be that $w_A = w_Y$, otherwise one of the sectors would not be able to employ people. This gives us the equation:

$$w_A = \delta A p_A = \frac{(1-\alpha)Y}{H_Y} = w_Y \tag{4}$$

Here we must impose a No Arbitrage Condition where the market price p_A of owning an idea should be equivalent to the integral of user cost of the idea (the cost of owning an idea for an instant and then selling it in the next instant). Also note that $K = xA$, so the user cost of an idea should be the same as the user cost of K. Formally, this means (using r to denote the user cost of an idea):

$$p_A = \int_0^\infty e^{-rt}\pi \, dt$$

But what is π? From equation (3), we know that π is constant on the BGP but it may not be constant elsewhere. So we need to discuss 2 cases:

- On the BGP (π is constant):

$$p_A = \int_0^\infty e^{-rt}\pi \, dt = \frac{\pi}{r} \Rightarrow rP_A = \pi$$

- On non-BGP (π is a function of t):

$$p_A(t) = \int_t^\infty e^{-\int_t^\infty r(s)\,ds}\pi(\tau)\,d\tau$$

$$\Rightarrow \dot{p_A}(t) = \underbrace{\int_t^\infty \pi(\tau) e^{-\int_t^\infty r(s)\,ds} \overbrace{\frac{\partial}{\partial t}\left[-\int_t^\tau r(s)\,ds\right]}^{=-r(t)} d\tau - \pi \cdot e^0}_{\text{Recall the Leibniz's Rule}}$$

$$= r(t) \underbrace{\int_t^{\infty} \pi(\tau) e^{-\int_t^{\infty} r(s)\,ds}\, d\tau}_{=p_A(t)} - \pi(t)$$

$$= r(t) p_A(t) - \pi(t)$$

$$\Rightarrow \ \pi(t) = r(t) p_A(t) - \dot{p_A}(t)$$

$$\Rightarrow \ r(t) p_A(t) = \pi(t) + \dot{p_A}(t)$$

Notice that just means that on the BGP, $\dot{p_A}(t) = 0$. So the no arbitrage condition is

$$g_{p_A} = \frac{\dot{p_A}}{p_A} = r - \frac{\pi}{p_A} \tag{5}$$

On the BGP, π is constant, and so is g_{p_A}, so p_A must be a constant and so on the BGP. $g_{p_A} = 0$. From equation (5), this means $p_A = \frac{\pi}{r} = \alpha(1-\alpha)\frac{Y}{rA}$ on the BGP.

Subbing equation (5) into equation (4), we know that:

$$\delta A p_A = \frac{(1-\alpha)Y}{H_Y} \ \Rightarrow \ \delta A \alpha(1-\alpha)\frac{Y}{rA} = \frac{(1-\alpha)Y}{H_Y} \ \Rightarrow \ r = \delta\alpha H_Y$$

$$\Rightarrow \ r = \delta\alpha(H - H_Y) = \delta\alpha H - \underbrace{\delta\alpha H_A}_{=\dot{A}} = \delta\alpha H - \alpha\frac{\dot{A}}{A}$$

$$\Rightarrow \ r = \alpha(\delta H - g_A) \tag{6}$$

Once again, r is the user cost of an idea. To write out r as a function of parameters and growth rates, we need to go back to the representative agent problem:

$$\max \int_0^{\infty} e^{-\rho t} u(c)\, dt \ s.t. \ \dot{k} = rk + w - c$$

Using the current value Hamiltonian, we have

$$\mathcal{H}^C = e^{-\rho t}\left\{\frac{c^{1-\sigma}-1}{1-\sigma} + \theta_1[rk + w - c]\right\}$$

The necessary conditions are

$$\frac{\partial \mathcal{H}^C}{\partial c} = 0 = c^{-\sigma} - \theta_1$$

$$\frac{\partial \mathcal{H}^C}{\partial k} = \rho\theta_1 - \dot{\theta}_1 = r\theta_1$$

The first condition gives us $g_{\theta_1} = -\sigma g_c$ and the second condition gibes us $\rho - g_{\theta_1} = r$. Combining these 2, we get:

$$r = \rho + \sigma g_c \tag{7}$$

Using equations (6) and (7), we have:

$$r = \delta\alpha H - \alpha g_A = \rho + \sigma g_c$$

On the BGP[29], $g_A = g_c$, so we have:

$$g_A = g_c = \frac{\delta\alpha H - \rho}{\sigma + \alpha} \tag{8}$$

Equation (8) tells us that the growth rate of this economy depends on the population H. If $H > \frac{\rho}{\alpha\delta}$, then the growth rate is positive. If $H < \frac{\rho}{\alpha\delta}$, then the growth rate is negative. If $H = \frac{\rho}{\alpha\delta}$, the economy would be stagnant. Hence our empirical question is to figure out why economic depressions can still happen when H is high, especially because by equation (8), if an economy's population grows, the growth rate should also grow.

To answer that question, we can relax our assumptions a little bit and suppose that there is externality to human capital investment (like in the Lucas model). In other words, we will change our assumption of $\dot{A} = \delta A H_A$ to

$$\dot{A} = \delta A^{\lambda} H_A^{\phi}$$

We can then derive the solutions like we just did, and make the model more flexible.

6.2.2 Social Planner's Problem

The social planner's problem is:

$$\max \int_0^{\infty} e^{-\rho t} \frac{C^{1-\sigma} - 1}{1-\sigma}\, dt$$

[29]On the BGP, $Y = (AH_Y)^{1-\alpha} K^{\alpha} \Rightarrow g_Y = (1-\alpha)g_A + \alpha g_K$. Since $MPK = \frac{\alpha Y}{K}$ is constant on the BGP¡ we must have $g_Y = g_K$, meaning $(1-\alpha)g_Y = (1-\alpha)g_A \Rightarrow g_Y = g_A$. As an economy, $\dot{K} = Y - C \Rightarrow g_K = \frac{Y}{K} - \frac{C}{K}$ is constant, so it must be that $g_Y = g_C$. So on the BGP, $g_Y = g_K = g_A = g_C$

$$s.t. \begin{cases} \dot{K} = (AH_Y)^{1-\alpha}K^{\alpha} - C \\ \dot{A} = \delta AH_A \end{cases}$$

and out goal is to compare g_A^{SP} to g_A^{GE}.

Using current value Hamiltonian, this system can be described by:

$$\mathcal{H}^C = e^{-\rho t}\left\{\frac{C^{1-\sigma}-1}{1-\sigma} + \theta_1\left[(AH_Y)^{1-\alpha}K^{\alpha} - C\right] + \theta_2[\delta AH_A]\right\}$$

Where $\{K, A\}$ are the state variables and $\{C, H_A\}$ are control variable.

The necessary conditions for optimal consumption and research are:

- $\dfrac{\partial \mathcal{H}^C}{\partial C} = 0 = C^{-\sigma} - \theta_1$, so $g_{\theta_1} = -\sigma g_C$
- $\dfrac{\partial \mathcal{H}^C}{\partial H_A} = 0 = \theta_1[(-1)(1-\alpha)(A)^{1-\alpha}H_Y^{-\alpha}K_{\alpha}] + \theta_2\delta A$

  From this condition, we get:

  $$\frac{\theta_1}{\theta_2} = \frac{\delta AH_Y}{(1-\alpha)Y}$$

  Log-linearize both sides and differentiate w.r.t. time, we get

  $$g_{\theta_1} = g_{\theta_2} + g_A - g_Y$$

- $\dfrac{\partial \mathcal{H}^C}{\partial K} = \rho\theta_1 - \dot{\theta}_1 = \theta_1[\alpha(AH_Y)^{1-\alpha}K^{\alpha-1}]$

  This means we have:

  $$\rho - g_{\theta_1} = \frac{\alpha Y}{K} = MPK$$

  So on the BGP, MPK is constant. Using log-linearization and differentiate w.r.t. time, we get that on the BGP: $g_Y = g_K$.

  Given the production function, we know that $g_Y = (1-\alpha)g_A + \alpha g_K \;\Rightarrow\; g_Y = g_A$

- $\dfrac{\partial \mathcal{H}^C}{\partial A} = \rho\theta_2 - \dot{\theta}_2 = \theta_1[(1-\alpha)A^{-\alpha}H_Y^{1-\alpha}K^\alpha] + \theta_2\delta H_A$

  This means we have:

$$\rho - g_{\theta_2} = -\underbrace{\frac{\delta A H_Y}{(1-\alpha)Y}}_{=\frac{\theta_1}{\theta_2}} \cdot \frac{(1-\alpha)Y}{A} + \delta H_A = \delta(H_Y + H_A) = \delta H$$

  From the constraint $\dot{K} = Y - C$, we know that $g_K = \frac{Y}{K} - \frac{C}{K}$. Since $g_K = g_Y$ and g_K is constant on the BGP, it must also be that $g_C = g_K$, and hence $g_C = g_Y = g_K = g_A$. Also, from $g_Y = g_A$, we know that $-\sigma g_C = g_{\theta_1} = g_{\theta_2} + g_A - g_Y = g_{\theta_2}$. So

$$\rho + \sigma g_C = \rho + \sigma g_A = \delta H$$

  This gives us the expression for g_A as:

$$g_A^{SP} = g_C^{SP} = \frac{\delta H - \rho}{\sigma}$$

Compare this with the general equilibrium results we have:

$$g_A^{SP} = g_C^{SP} = \frac{\delta H - \rho}{\sigma} > \frac{\delta \alpha H - \rho}{\sigma + \alpha} = g_A^{GE} = g_c^{CE}$$

This implies that the imperfect competition in the intermediate goods market (in the GE environment) guarantees R&D (otherwise only invest in capital) but leads to a dead-weight-loss on productivity.

7 Business Cycle Theory

Now that we have studied growth of the economy (τ_t) for a while, it's time for us to study the deviations from growth (d_t). As we have learned in the last two sections, without uncertainty, our economy should be able to grow steadily (i.e., Balanced Growth Path). But the reality is that the world simply does not work that way. In an attempt to study why that is the case, economists have tried to model the uncertainties in the economy as exogenous shocks, on the demand and/or on the supply.

Generally, the **demand shocks** are what we have been learning/are used to seeing. This includes mostly Keynesian models as well as Lucas's models on the representative agents expectations (such as sudden change in monetary policies). On the other hand, the studies of **supply shocks** takes a more classical approach and leads to the **Efficiency Wage Theory** and what is commonly understood as the **Real Business Cycle (RBC)** where the existence of money is redundant[30] in the competitive market.

7.1 Efficiency Wage Theory

Our first attempt at studying supply shocks is simple. We know that firms in the economy choose employment and, through that, wages (when labor market clears). Efficiency wage theory posits that workers' efforts/productivity can also be thought of as a function of the real wage. So instead of having a unique wage at each labor market-clearing quantity, there can be a continuum of market-clearing wages that increases along with productivity (think of it as revere causality, if you wish).

For example, let w denote the nominal wage and $E(w)$ denote each identical worker's "effort" when wage w is paid to workers. As such, part of the firms' profit maximization problem is:

$$\max_{H,w} \quad \underbrace{PE(w)H}_{\text{Total marginal revenue from labor}} \quad - \quad \underbrace{wH}_{\text{Cost of labor}}$$

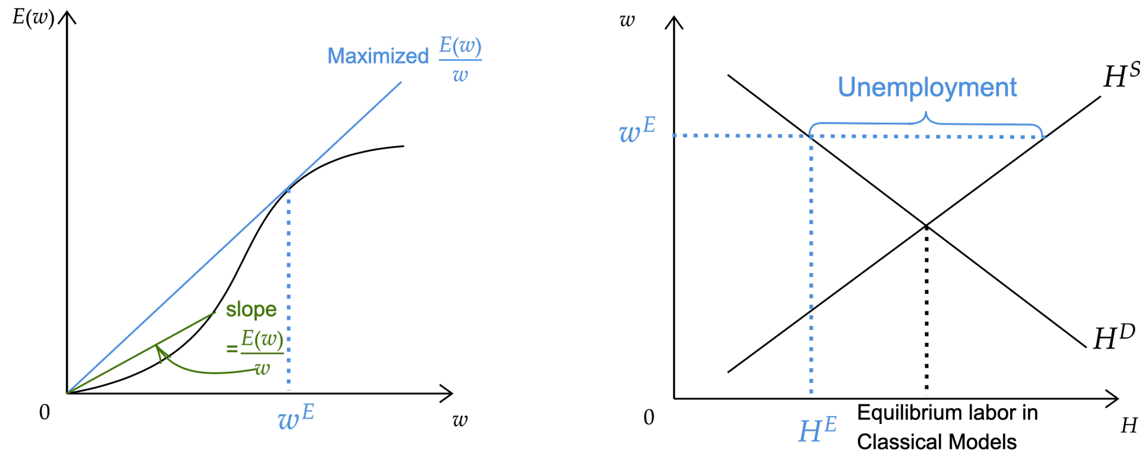
The F.O.C.s are:

$$\frac{\partial \mathcal{L}}{\partial w} : E'(w)H = H \;\Rightarrow\; E'(w) > 0$$

[30]Because equilibria are in real terms. Recall that firms solve a profit maximization problem that is, in principle, arbitrarily scalable in the long run.

$$\frac{\partial \mathcal{L}}{\partial H} : E(w) = w \;\Rightarrow\; E'(w) = 1 \;\Rightarrow\; \frac{E'(w)}{E/w} = 1$$

The figures below illustrates how the equilibrium would be different than the classical approach:



7.2 Real Business Cycle Theory

Another approach to study supply shocks is through *Real Business Cycle Theory*, where every thing is dealt with in real terms.

7.2.1 Social Planner's Problem

The social planner's problem in the economy, that we all know and love, is:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t) \; s.t. \; c_t = f(k_t, h_t) - k_{t+1} + k_t(1-\delta)$$

From here, we have 2 approaches to the problem. We can either solve it with the *Dynamic Programming Approach* or the *Calibration Approach*.

Dynamic Programming Approach to Real Business Cycle Theory:
We have the state variable $\{k_t\}$ and the control variables $\{k_{t+1}, h_t\}$ and the Bellman equation is:

$$V(k_t) = \max_{k_{t+1}, h_t} \{u(c_t, 1-h_t) + \beta V(k_{t+1})\}$$

So the F.O.C.s are:

$$[k_{t+1}]: \; -u_c(c_t, 1-h_t) + \beta V_{k_{t+1}} = 0$$
$$[h_t]: u_c(c_t, 1-h_t) \cdot f_h(k_t, h_t) - u_h(c_t, 1-h_t) = 0 \;\Rightarrow\; [ECL]$$

Using these F.O.C.s, we get the optimal decision rules $k_{t+1} = k(k_t)$, $h_t = h(k_t)$.
We then differentiate the Bellman equation with respect to k_t and get

$$V_{k_t} = u_c(c_t, 1-h_t)[f_k(k_t, h_t) + (1-\delta)]$$
$$\Rightarrow V_{k_{t+1}} = u_c(c_{t+1}, 1-h_{t+1})[f_k(k_{t+1}, h_{t+1}) + (1-\delta)]$$

Substituting in the optimal decision rules, we get

$$u_c(c_t, 1-h_t) = \beta u_c(c_{t+1}, 1-h_{t+1})[f_k(k_{t+1}, h_{t+1} + (1-\delta] \; [ECC]$$

As such, our equilibrium is characterized by:

(i) $u_c(c_t, 1-h_t) = \beta u_c(c_{t+1}, 1-h_{t+1})[f_k(k_{t+1}, h_{t+1}) + (1-\delta)]$

(ii) $u_h(c_t, 1-h_t) = u_c(c_t, 1-h_t) f_h(k_t, h_t)$

(iii) k_0 is given

(iv) Transversality Condition $\left(\lim_{t\to\infty} \beta^t u_c(c_t, 1-h_t) k_{t+1} = 0\right)$

Calibration Approach:
Suppose that the agents in the economy have utility function $u(c, h) = ln(c) + Aln(1-h)$ and the production technology is $f(k, h) = k^\theta h^{1-\theta}$ (Cobb-Douglas). We want to use real-world data to "calibrate" for A, θ, and δ.

Let $y_t = z_t f(k_t, h_t)$ where z_t is the total factor productivity that is susceptible to exogenous shocks. We can write z_t as:

$$z_t = \frac{y_t}{k_t^\theta h_t^{1-\theta}}$$

Suppose that z_t follows a Markov process such that $z_{t+1} = z(z_t, \nu_t)$. The social planner's problem in this economy is thus:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t) \ s.t. \ c_t = z_t f(k_t, h_t) - k_{t+1} + (1-\delta)k_t$$

with state variables $\{k_t, z_t\}$ and control variables $\{k_{t+1}, h_t\}$ so the Bellman equation is:

$$V(k_t, \boldsymbol{z_t}) = \max_{k_{t+1}, h_t} \{u(c_t, 1-h_t) + \beta E\left[V(k_{t+1}, \boldsymbol{z_{t+1}})\right]\}$$

Using the same calculations you should be familiar with, we derive the equilibrium paths to be:

(i) $k_{t+1} = k(k_t, z_t)$, $h_t = h(k_t, z_t)$

(ii) $u_c(c_t, 1-h_t) = \beta E\left[u_c(c_{t+1}, 1-h_{t+1})[z_{t+1} f_k(k_{t+1}, h_{t+1}) + (1-\delta)]\right]$

(iii) $u_h(c_t, 1-h_t) = u_c(c_t, 1-h_t) z_t f_h(k_t, h_t)$

(iv) k_0, z_0 is given

(v) Transversality Condition $\left(\lim_{t\to\infty} \beta^t u_c(c_t, 1-h_t) k_{t+1} = 0\right)$

This means that, in steady-state, we have

$$1 = \beta[f_k(k^*, h^*) + (1-\delta)] \ \Rightarrow \ 1+\rho = \underbrace{\theta \frac{y^*}{k^*}}_{f(k,h)=k^\theta h^{1-\theta}} + 1 - \delta \ \Rightarrow \ \rho + \delta = \theta \frac{y^*}{k^*}$$

Empirically , $\rho \approx 0.05$, $\delta \approx 0.1$, $\theta \approx 0.4 \ \Rightarrow \ \frac{k^*}{y^*} \approx 2$ (so this model is consistent with the real world).

From here, we get

$$\frac{1}{c^*}(1-\theta)\frac{y^*}{h^*} = \frac{A}{1-h^*} \ \Rightarrow \ (1-\theta)\frac{y^*}{h^*} = A\frac{c^*}{1-h^*} = A\frac{y^* - \delta k^*}{1-h^*}$$

Empirically, $h^* \approx \frac{1}{3}$ and $A \approx 1.75$.

7.2.2 Recursive Competitive Equilibrium Problem without Distortions

Recall the simple set up of a 1-period model:

The representative agents solve the problem:

$$\max u(c, 1-h) \ s.t. \ c = wh + rk \ \Rightarrow \ \frac{u_h}{u_c} = w$$

The representative firm's problem is:

$$\max_{k,h} \pi = f(k,h) - rk - wh \ \Rightarrow \ f_k = r, \ f_h = w$$

Now, we will expand this into the infinite horizon model with uncertainty where

- $t = 0, 1, 2, \ldots, \infty$
- $z_{t+1} = z(z_t, \varepsilon_t)$ (shocks to productivity). Let $ln(z_{t+1}) = \mu ln(z_t) + \varepsilon_t$, $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$
- $Y_t = z_t F(K_t, H_t)$
- Exogenous law of motion

The firm solves the problem:

$$\max_{K_t, H_t} z_t F(K_t, H_t) - w_t H_t - r_t K_t$$

So the F.O.C.s are:

$$w_t = z_t F_H(K_t, H_t)$$
$$r_t = z_t F_K(K_t, H_t)$$

So the exogenous wages and rent are determined by aggregate capital and labor in each period. This means that agents must have expectations about the aggregate, but they don't know how their choices will actually affect the wages and rents they receive.

The representative agent then solves the problem:

$$\max_{\{k_{t+1}, h_t\}} E\left[\sum_{t=0}^{\infty} u(c_t, 1-h_t)\right] \ s.t. \ c_t = w_t h_t + r_t k_t - k_{t+1} + k_t(1-\delta)$$

As such, we must have the assumption of **Rational Expectation** - Agents correctly predict the "law of motion" of aggregate variables. Formally, this means

$$K_{t+1} = K(K_t, z_t), \ H_t = H(K_t, z_t)$$

We must have state variables $\{k_t, \boldsymbol{K_t}, z_t\}$ and control variables $\{k_{t+1}, h_t\}$, giving use the Bellman Equation:

$$V(k_t, K_t, z_t) = \max_{\{k_{t+1}, h_t\}} \{u(c_t, 1-h_t) + \beta E[V(k_{t+1}, K_{t+1}, z_{t+1})]\}$$

We can thus characterize the equilibrium with:

- $u_c(c_t, 1-h_t) = \beta E[u_c(c_{t+1}, 1-h_{t+1})[r_{t+1} + 1 - \delta]]$
- $u_c(c_t, 1-h_t)w_t = u_h(c_t, 1-h_t)$
- k_0, z_0 are given
- TVC
- (Market Clearing) $k_{t+1} = k(k_t, K_t, z_t)$
- (Consistency) $K(k_t, K_t, z_t) = \underbrace{K(K_t, K_t, z_t)}_{\text{Optimal Decision}} = \underbrace{K(K_t, z_t)}_{LOMOC}$

The caveat here is that if there are N identical agents (instead of measure 1), the market clearing condition is $N \cdot k_t = K_t$ and the consistency condition is $N \cdot k\left(\frac{K_t}{N}, K_t, z_t\right) = K(K_t, z_t)$.

Classical results suggests that, under no distortionary tax, the Competitive Equilibrium is the same as the Pareto Equilibrium such that:

$$c_t^{CE} = r_t k_t + w_t h_t - k_{t+1} + k_t(1-\delta) = \underbrace{z_t f_k(K_t, H_t)k_t + z_t F_H(K_t, H_t)h_t}_{=y_t \text{ under linear technology}} - i_t = y_t - i_t = c_t^{SP}$$

So one simpler approach in solving a non-distortionary CE is to solve the SP problem then calculate w_t, r_t using the firm's problem.

Definition (RCE): In a dynamic optimization problem with uncertainty, a **Recursive Competitive Equilibrium** is a list $V(k_t, K_t, z_t)$, $k_{t+1} = k(k_t, K_t, z_t)$, $h_t = h(k_t, K_t, z_t)$, $k_{t+1} = K(K_t, z_t)$ such that

- RAs maximize utility given w_t, r_t
- RFs maximize profit given w_t, r_t
- Markets clear $k_{t+1} = k(k_t, K_t, z_t)$, $h_t = h(k_t, K_t, z_t)$
- Consistency $K_{t+1} = K(k_t, K_t, z_t) = K(K_t, z_t)$

7.2.3 Recursive Competitive Equilibrium with Distortions

Consider out last environment, but now with the inclusion of a government with spending g_t. For now, we will assume that government spending is an exogenous stochastic variable such that $g_{t+1} = G(g_t, \mu_t)$.

We know from before, that the government in the **closed economy** must balance its budget with taxes τ_k, τ_w (distortionary) and T_t (lump-sum) such that $g_t = \tau_h w_t h_t + \tau_k r_t k_t + T_t$.

In this environment, the social planner's problem is:

$$\max \sum_{t=0}^{\infty} \beta E[u(c_t, 1-h_t)] \ s.t. \ c_t = z_t f(k_t, h_t) - k_{t+1} + k_t(1-\delta) - g_t$$

We have the state variables[31] $\{k_t, z_t, g_t\}$ and control variables $\{k_{t+1}, h_t\}$.

The Bellman equation is thus:

$$V(k_t, z_t, g_t) = \max_{k_{t+1}, h_t} \{u(c_t, 1-h_t) + \beta E[V(k_{t+1}, z_{t+1}, g_{t+1})]\}$$

The Pareto equilibrium is thus characterized by

- $u_c(c_t, 1-h_t) = \beta E[u_c(c_{t+1}, 1-h_{t+1})[r_{t+1} + 1 - \delta]]$
- $u_c(c_t, 1-h_t) z_t f_h(k_t, h_t) = u_h(c_t, 1-h_t)$
- k_0, z_0, g_0 are given
- TVC

To solve for the *RCE*, we solve for the representative agent's problem first

$$\max \sum_{t=0}^{\infty} \beta E[u(c_t, 1-h_t)] \ s.t. \ c_t = (1-\tau_w) w_t h_t + (1-\tau_k) r_t k_t - k_{t+1} + k_t(1-\delta) - T_t$$

Given LOMOC $K_{t+1} = K(K_t, z_t, g_t)$, we have the state variables $\{k_t, K_t, z_t, g_t\}$ and control variables $\{k_t, h_t\}$. The resulting Bellman equation is:

$$V(k_t, K_t, z_t, g_t) = \max_{\{k_{t+1}, h_t\}} \{u(c_t, 1-h_t) + \beta E[V(k_{t+1}, K_{t+1}, z_{t+1}, g_{t+1})]\}$$

[31] For the social planner, consistency and market clearing will hold true automatically, so the aggregate variables can be omitted here

The Euler conditions are thus:

$$u_c(c_t, 1-h_t) = \beta E[u_c(c_{t+1}, 1-h_{t+1})[(1-\tau_k)r_t + k_{t+1} + 1 - \delta]]$$
$$u_c(c_t, 1-h_t)(1-\tau_h)w_t = u_h(c_t, 1-h_t)$$

Next, we solve the representative firm's problem:

$$\max z_t F(K_t, H_t) - w_t H_t - r_t K_t$$

so the F.O.C.s are

$$r_t = z_t F_K(K_t, H_t)$$
$$w_t = z_t F_H(K_t, H_t)$$

We can then characterize the *Recursive Competitive Equilibrium* as a list of $V(k_t, K_t, z_t, g_t)$, $k_{t+1} = k(k_t, K_t, z_t, g_t)$, $h_t = h(k_t, K_t, z_t, g_t)$, K_{t+1}, and $\{T_t\}$ such that

- RAs maximize utility given $\{w_t, r_t\}$, and $\{T_t\}$
- RFs maximize profit given r_t, w_t
- Markets clear $k_{t+1} = k(k_t, K_t, z_t)$, $h_t = h(k_t, K_t, z_t)$
- Consistency $K_{t+1} = K(k_t, K_t, z_t) = K(K_t, z_t)$
- Government budget is balanced $y_t = \tau_h w_t h_t + \tau_k r_t k_t + T_t$

8 Overlapping Generations Model

8.1 Basic OLG

The Basic model has the following environment:

- $t = 0, 1, 2, \ldots$. In $t = 0$, a generation of old people is present.
- In each generation (including $t = 0$), a new generation of measure 1 population is born. Denote generation t as G_t
- In each generation $t > 0$, the old generation in $t-1$ dies.
- Each new generation has endowments (e_1, e_2). e_1 for when they are young and e_2 for when they are old.
- Consumption of generation t is denoted $(c_{t,1}, c_{t,2})$ where the second subscript indicates consumption when young (1) or old (2).

Consider the following two equilibrium concepts:

A **Recursive Competitive Equilibrium** is a list $(p_t, c_{t,1}, c_{t,2})$ such that

- $C_{0,2} = e_2$
- $\forall G_t, t \geq 1$, the list is the solution to

$$\max\ u(c_{t,1}, c_{t,2})$$
$$s.t.\ \begin{cases} c_{t,1} = e_1 - s_t \\ c_{t,2} = e_2 + R_t s_t \end{cases}$$

  where R_t is the gross interest rate of savings.

- $c_{t,1} + c_{t-1,2} = e_1 + e_2$ (feasibility)

Using DP, we can characterize the RCE with:

$$\max\ \ u(e_1 - s_t, e_2 + R_t s_t)$$

We get the F.O.C.s and get:

$$[s_t]\ \ u_1(c_{t,1}, c_{t,2}) = R_t u_2(c_{t,1}, c_{t,2})$$
$$\Rightarrow\ \underbrace{\mu(c_{t,1}, c_{t_2})}_{\text{MRS}} = \frac{u_1(c_{t,1}, c_{t,2})}{u_2(c_{t,1}, c_{t,2})} = R_t$$

A **Walrasian Competitive Equilibrium** is a list $(p_t, c_{t,1}, c_{t,2})$ such that

- $C_{0,2} = e_2$
- $\forall G_t, t \geq 1$, the list is the solution to

$$\max\ u(c_{t,1}, c_{t,2})$$
$$s.t.\ p_t c_{t,1} + p_{t+1} c_{t,2} = p_t e_1 + p_{t+1} e_2$$

- $c_{t,1} + c_{t-1,2} = e_1 + e_2$ (feasibility)

Using Lagrangian, we characterize the WEA with:

$$\mathcal{L} = u(c_{t,1}, c_{t,2}) + \lambda[p_t c_{t,1} + p_{t+1} c_{t,2} - p_t e_1 - p_{t+1} e_2]$$

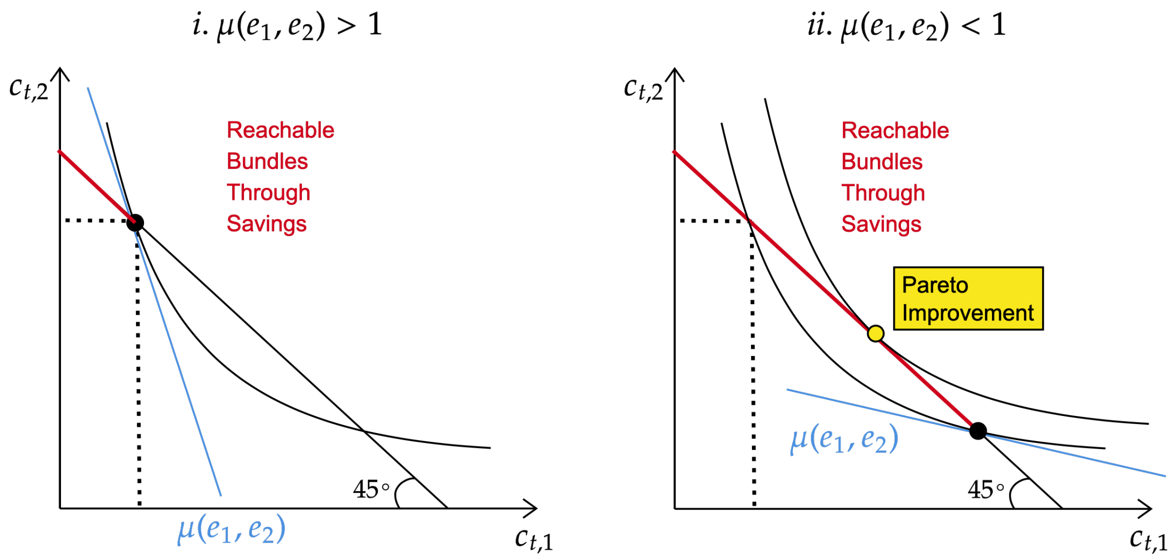
We get the F.O.C.s and get:

$$\begin{Bmatrix} [c_{t,1}] & u_1(c_{t,1}, c_{t,2}) = \lambda p_t \\ [c_{t,2}] & u_2(c_{t,1}, c_{t,2}) = \lambda p_{t+1} \end{Bmatrix}$$
$$\Rightarrow \frac{u_1(c_{t,1}, c_{t,2})}{u_2(c_{t,1}, c_{t,2})} = \frac{p_t}{p_{t+1}}$$

So we know (WCE $\iff$ RCE) $\iff$ $(\frac{p_t}{p_{t+1}} = R_t)$

Results:

1. The only equilibrium is **Autarky**[32] so $(c_{t,1}, c_{t,2}) = (e_1, e_2)$.
2. The equilibrium allocation is not Pareto Optimal.
   Consider $(e_1, e_2) = (1, 0)$. The consumption $(0, 1)$ for all generations Pareto dominates the equilibrium Autarky $(1, 0)$.
3. Autarky allocation is efficient if and only if $\mu(e_1, e_2) \geq 1$. The graphs below should suffice as proof (If discount is β, then it's $\mu \geq \beta$.



8.2 OLG with Population Growth

Consider the same environment as the basic model, except that the population that gets born grows at a gross rate $\gamma \equiv \frac{N_{t+1}}{N_t}$. This means that the feasibility condition is now:

$$
\begin{aligned}
N_t c_{t,1} + N_{t-1} c_{t-1,2} &= N_t e_1 + N_{t-1} e_2 \\
\Rightarrow \ \frac{N_t}{N_{t-1}} c_{t,1} + c_{t-1,2} &= \frac{N_t}{N_{t-1}} e_1 + e_2 \\
\Rightarrow \ \gamma c_{t,1} + c_{t-1,2} &= \gamma e_1 + e_2
\end{aligned}
$$

Similar to the basic model, the result is that Autarky is Pareto efficient if and only if $\mu(e_1, e_2) \geq \gamma$.

[32]Agents consume only endowments.

8.3 OLG with Money

Consider the same environment as the basic model except with add in:

- M: Supply of money that is given to the initial old generation.
- q_t: Price of money holdings ($q_t = \frac{1}{p_t}$ where p_t is the price of goods if goods have prices).

Similar to the basic case we will consider the following two equilibrium concepts:

A **Recursive Competitive Equilibrium** is a list $(p_t, q_t, R_t, c_{t,1}, c_{t,2})$ such that

- $C_{0,2} = e_2 + q_1 M$
- $\forall G_t, t \geq 1$, the list is the solution to

$$\max u(c_{t,1}, c_{t,2})$$
$$s.t. \begin{cases} c_{t,1} = e_1 - q_t \cdot \underbrace{m_t}_{\substack{\text{Demand}\\ \text{for}\\ \text{money}}} - s_t \\ c_{t,2} = e_2 + q_{t+1} m_t + R_t s_t \end{cases}$$

where R_t is the gross interest rate of savings.

- $c_{t,1} + c_{t-1,2} = e_1 + e_2$ (feasibility)

The savings rate in this system is determined by:

$$\begin{aligned} & s_t = e_1 - q_t m_t - c_{t,1} \\ \Rightarrow\ & c_{t,2} - e_2 = q_{t+1} m_t + R_t(e_1 - q_t m_t - c_{t,1}) \\ \Rightarrow\ & (c_{t,2} - e_2) + R_t(c_{t,1} - e_1) = q_{t+1} m_t - R_t q_t m_t \\ \Rightarrow\ & (c_{t,2} - e_2) + R_t(c_{t,1} - e_1) = q_t m_t \left(\frac{q_{t+1}}{q_t} - R_t \right) \end{aligned}$$

In a monetary equilibrium, we must have $\frac{q_{t+1}}{q_t} = R_t$ in order to have $m_t \neq 0$, which means the savings mechanism is redundant.

A **Walrasian Competitive Equilibrium** is a list $(p_t, c_{t,1}, c_{t,2})$ such that

- $C_{0,2} = e_2 + q_1 M$
- $\forall G_t, t \geq 1$, the list is the solution to

$$\max u(c_{t,1}, c_{t,2})$$
$$s.t.\ p_t c_{t,1} + p_{t+1} c_{t,2} = p_t e_1 + p_{t+1} e_2$$

- $c_{t,1} + c_{t-1,2} = e_1 + e_2$ (feasibility)

Using Lagrangian, we characterize the WEA with:

$$\mathcal{L} = u(c_{t,1}, c_{t,2}) + \lambda[p_t c_{t,1} + p_{t+1} c_{t,2} - p_t e_1 - p_{t+1} e_2]$$

We get the F.O.C.s and get:

$$\begin{Bmatrix} [c_{t,1}] & u_1(c_{t,1}, c_{t,2}) = \lambda p_t \\ [c_{t,2}] & u_2(c_{t,1}, c_{t,2}) = \lambda p_{t+1} \end{Bmatrix}$$
$$\Rightarrow \frac{u_1(c_{t,1}, c_{t,2})}{u_2(c_{t,1}, c_{t,2})} = \frac{p_t}{p_{t+1}} = \frac{q_{t+1}}{q_t}$$

To define the monetary equilibrium, we must rewrite the utility maximization problem as:

$$\max u(c_{t,1}, c_{t,2})$$
$$s.t. \begin{cases} c_{t,1} = e_1 - q_t m_t \\ c_{t,2} = e_2 + q_{t+1} m_t \end{cases}$$

The F.O.C is then

$$\begin{aligned}
&[\partial m_t]\ q_t u_1(c_{t,1}, c_{t,2}) = q_{t+1} u_2(c_{t,1}, c_{t,2}) \\
&\quad \Rightarrow \mu(c_{t,1}, c_{t,2}) \equiv \frac{u_1(c_{t,1}, c_{t,2})}{u_2(c_{t,1}, c_{t,2})} = \frac{q_{t+1}}{q_t} \\
&\quad \Rightarrow \mu(e_1 - q_t m_t, e_2 + q_{t+1} m_t) = \frac{q_{t+1}}{q_t} = \frac{p_t}{p_{t+1}} \\
&\quad \Rightarrow q_{t+1} = q_t \cdot \mu(e_1 - q_t m_t, e_2 + q_{t+1} m_t) \text{ (Optimal rule for price of money)}
\end{aligned}$$

and the money market clearing condition is $m_t = M$.

Definition (ME): A **Monetary Equilibrium** is a sequence of $\{q_t\}$ such that

- $\frac{q_{t+1}}{q_t} = \mu(e_1 - q_t m_t, e_2 + q_{t+1} m_t)$
- $\{q_t\}$ is a bounded sequence

In a steady-state monetary equilibrium, we must then have $q_t = q_{t+1} = q^*$ and

$$s^* = q^* M$$
$$\mu(e_1 - s^*, e_2 + s^*) = 1$$

and $f(0) = 0$, $f'(q_t) \mid_{q_t=0} = \mu(e_1, e_2)$. To find the steady-state, we define an implicit function

$$T(q) = -u_1(e_1 - qM, e_2 + qM) + u_2(e_1 - qM, e_2 + qM)$$

where in steady-state q^* we have:

$$T(q^*) = 0 \iff \mu(e_1 - q^* M, e_2 + q^* M) = 1 \iff q^* = f(q^*)$$

Can such q^* actually exist though?

$$\frac{\partial T}{\partial q} = u_{11} M - u_{12} M - u_{21} M + u_{22} M = M(\underbrace{u_1 1}_{<0} - 2\underbrace{u_{12}}_{>0} + \underbrace{u_{22}}_{<0}) < 0$$

So T is a strictly decreasing function in q and so a unique steady-state exists if and only if $T(q_0) > 0$. Notice that

$$T(q_0) > 0 \iff -u_1 + u_2 > 0 \iff \mu(q_0) < 1 \iff \mu(e_1, e_2) < 1$$

This result reaffirms our previous results that if $\mu(e_1, e_2) > 1$, then Autarky is Pareto Optimal, but if $\mu(e_1, e_2) < 1$, then a monetary equilibrium is Pareto Optimal. As such, the existence of money in equilibrium is merely a means to an end, given that the non-monetary sector was inefficient to begin with.

Example:

Suppose that the representative agents' maximization problem can be characterized with:

$$\max\ ln(c_{t,1}) + \beta ln(c_{t,2})\ s.t. \begin{cases} c_{t,1} = e_1 - q_t m_t \\ c_{t,2} = e_2 + q_{t+1} m_t \end{cases}$$

The F.O.C. is

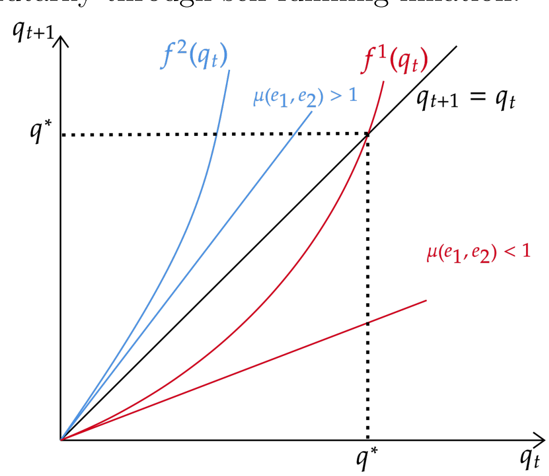
$$\frac{q_t}{e_1 - q_t m_t} = \frac{\beta q_{t+1}}{e_2 + q_{t+1} m_t} \Rightarrow m_t = \frac{\beta e_1 q_{t+1} - e_2 q_t}{(1+\beta) q_t q_{t+1}} \Rightarrow m^* = \frac{\beta e_1 - e_2}{(1+\beta) q^*}$$

So $m^* > 0$ in equilibrium if and only if $\beta e_1 > e_2$ which is equivalent to $\mu(e_1, e_2) < 1$. The optimal rule for the price of money $q_{t+1} = f(q_t) = \frac{e_2 q_t}{\beta e_1 - (1+\beta) q_t M}$ must satisfy:

- $f(0) = 0$
- $f'(0) = \frac{e_2[\beta e_1 - (1+\beta) q_t M] + e_2 q_t (1+\beta) M}{[\beta e_1 - (1+\beta) q_t M]^2}\Bigg|_{q_t=0} = \frac{e_2}{\beta e_1} = \mu(e_1, e_2)$
- $f'(q_t) = \frac{\beta e_1 e_2}{[\beta e_1 - (1+\beta) q_t M]^2} > 0,\ f''(q_t) > 0$

Consider the following 2 cases f^1 and f^2. In f^1, a monetary equilibrium exists at q^*, but in f^2, the function becomes unbounded (as it only intersects the 45-degree line at 0), so no monetary steady-state exists.

Autarky through self-fulfilling inflation.



The result is that if $\mu(e_1, e_2) < 1$, then a steady-state monetary equilibrium is unique and there are infinitely many paths given $q_0 < q^*$. If $\mu(e_1, e_2) > 1$, the dynamic equilibrium for $q_0 < q^*$ converges to

8.4 OLG with Changes in Money Supply

Consider the same environment as in OLG with money, but now there is a government with exogenous supply of fiat money such that

- $\dfrac{M_{t+1}}{M_t} = z$
- The old generation in period t receives the changed money supply

This can be done in a couple of ways:

8.4.1 Lump-Sum Transfers

In this case, agents solve the problem:

$$\max \ u(c_{t,1}, c_{t,2}) \ s.t. \begin{cases} c_{t,1} = e_1 - q_t m_t \\ c_{t,2} = e_2 + q_{t+1}(m_t + \tau_t) \end{cases}$$

where τ_t is the increase in the money supply so $m_t + \tau_t = M_{t+1}$ for the market to clear. Using F.O.C. with respect to m_t, we get

$$\frac{q_{t+1}}{q_t} = \mu(e_1 - q_t M_t, e_2 + q_{t+1} M_{t+1})$$

Using the same logic that money is an equivalent saving mechanism as s_t, we get that

$$\frac{s_{t+1}}{s_t} \frac{M_t}{M_{t+1}} = \frac{s_{t+1}}{s_t} \frac{1}{z} = \mu(e_1 - s_t, e_2 + s_{t+1})$$

In this case, money is neutral but not super-neutral since money supply M_t itself does not affect consumption smoothing, but the **growth rate z** affects consumption smoothing.

Example:

Suppose that the representative agents' maximization problem can be characterized with:

$$\max \ ln(c_{t,1}) + \beta ln(c_{t,2}) \ s.t. \begin{cases} c_{t,1} = e_1 - q_t m_t \\ c_{t,2} = e_2 + q_{t+1}(m_t + \tau_t) \end{cases}$$

The F.O.C. of this problem is

$$\frac{q_t}{e_1 - q_t m_t} = \frac{\beta q_{t+1}}{e_2 + q_{t+1}(m_t + \tau_t)}$$

Plugging in the market clearing condition ($M_t = m_t$, $M_{t+1} = m_t + \tau_t$), we have

$$\frac{q_t M_t}{e_1 - q_t M_t} = \frac{\beta q_{t+1} M_t}{e_2 + q_{t+1} M_{t+1}} = \frac{\beta q_{t+1} M_{t+1}/z}{e_2 + q_{t+1} M_{t+1}} \Rightarrow \frac{s_t \cdot z}{e_1 - s_t} = \frac{\beta s_{t+1}}{e_2 + s_{t+1}} \quad (\star)$$

In steady-state, we have $s_t = s_{t+1} = s^*$ so

$$\frac{z}{e_1 - s^*} = \frac{\beta}{e_2 + s^*} \Rightarrow \frac{e_2 + s^*}{\beta(e_1 - s^*)} = \mu(e_1 - s^*, e_2 + s^*) = \frac{1}{z} \Rightarrow s^* = \frac{\beta e_1 - z e_2}{z + \beta}$$

Like before, this means that $s^* > 0$ if and only if $\beta e_1 - z_2 > 0$, which is equivalent to $z\mu(e_1, e_2) < 1$. The optimal savings rule is thus:

$$s_{t+1} = f(s_t) = \frac{e_2 z s_t}{\beta e_1 - (\beta + z)s_t}$$

and it must satisfy:

- $f(0) = 0$
- $f'(s_t) = \dfrac{e_2 z[\beta e_1 - (\beta + z)s_t] + e_2 z s_t(\beta + z)}{[\beta e_1 - (\beta + z)s_t]^2} = \dfrac{\beta e_1 e_2 z}{[\beta e_1 - (\beta + z)s_t]^2} > 0$
- $f'(0) = \frac{\beta e_1 e_2 z}{\beta^2 e_1^2} = z\mu(e_1, e_2)$
- $f''(s_t) = \dfrac{2[\beta e_1 - (\beta + z)s_t](\beta + z)}{[\beta e_1 - (\beta + z)s_t]^4} > 0$

The result is similar to the pure money model where if $z\mu(e_1, e_2) < 1$, then a unique steady-state monetary equilibrium exists where $s* = q^* M^*$, so the price of money must grow at the same rate as the money growth rate.

The next question then is to see if there is such thing as an "optimal level" for the money growth rate z where optimal means overall welfare maximizing. In equilibrium, we know that

$$s^* = \frac{\beta e_1 - z e_2}{z + \beta}$$

so $\forall G_t$, $t \geq 1$, we must have

$$\begin{aligned}
u(c_1^*, c_2^*) &= ln(e_1 - s^*) + \beta ln(e_2 + s^*) = ln\left(e_1 - \frac{\beta e_1 - ze_2}{z+\beta}\right) + \beta ln\left(e_2 + \frac{\beta e_1 - ze_2}{z+\beta}\right) \\
&= ln\left(e_1 - \frac{z(e_1+e_2)}{z+\beta}\right) + \beta ln\left(e_2 + \frac{\beta(e_1+e_2)}{z+\beta}\right) \\
&= ln(z) - (1+\beta)ln(z+\beta) + C
\end{aligned}$$

The F.O.C. with respect to z is thus:

$$\frac{1}{z} - \frac{1+\beta}{z+\beta} = 0 \;\Rightarrow\; z^* = 1, \forall G_t,\, t \geq 1$$

But what about the old generation at $t = 0$? We know that in this system they have

$$u(c_{0,2}) = ln(c_{0,2}) = ln\left(\frac{\beta(e_2)}{z+\beta}\right)$$

So the initial old generation's utility is decreasing in z. As such, any $z \leq 1$ Pareto dominates $z > 1$.

8.4.2 Proportional Money Transfer

In this case, agents solve the problem:

$$\max\; u(c_{t,1}, c_{t,2})\; s.t. \begin{cases} c_{t,1} = e_1 - q_t m_t \\ c_{t,2} = e_2 + q_{t+1} m_t \cdot z \end{cases}$$

The F.O.C. of this problem is thus:

$$\begin{aligned}
\frac{q_{t+1}\cdot z}{q_t} = \frac{u_1}{u_2} &= \mu(c_{t,1}, c_{t,2}) = \mu(e_1 - q_t M_t, e_2 + q_{t+1} M_{t+1}) \\
\Rightarrow\; \frac{M_{t+1} q_{t+1} z}{M_t q_t} \frac{M_t}{M_{t+1}} &= \mu(e_1 - q_t M_t, e_2 + q_{t+1} M_{t+1}) \\
\Rightarrow\; \frac{s_{t+1}\cdot z}{s_t}\frac{1}{z} = \frac{s_{t+1}}{s_t} &= \mu(e_1 - s_t, e_2 + s_{t+1})
\end{aligned}$$

This means that money is both neutral and super neutral, as the growth rate of money z does not affect consumption smoothing.

8.4.3 Inflation Tax: Government with Non-Fiat Money Supply

In this case, agents solve the problem:

$$\max\ u(c_{t,1}, c_{t,2})\ s.t.\ \begin{cases} c_{t,1} = e_1 - q_t m_t \\ c_{t,2} = e_2 + q_{t+1} m_t \end{cases}$$

But the market clearing conditions now include:

- $M_{t+1} = M_t + \underbrace{(M_{t+1} - M_t)}_{\substack{\text{Changes in} \\ \text{money supply}}}$
- Government balances budget $g_t = q_t(M_{t+1} - M_t)$

The representative agent's problem F.O.C. is:

$$\begin{aligned} q_t u_1 = q_{t+1} u_2 \ &\Rightarrow \mu(c_{t,1}, c_{t,2}) = \frac{u_1}{u_2} = \frac{q_{t+1}}{q_t} \underbrace{= \mu(e_1 - q_t M_t, e_2 + q_{t+1} M_t)}_{\text{Market Clearing}} \\ &\Rightarrow \mu\left(e_1 - q_t M_t, e_2 + q_{t+1} M_{t+1} \frac{M_t}{M_{t+1}}\right) = \frac{q_{t+1} M_{t+1}}{q_t M_t} \frac{M_t}{M_{t+1}} \\ &\Rightarrow \mu\left(e_1 - s_t, e_2 + s_{t+1} \frac{1}{z}\right) = \frac{s_{t+1}}{s_t} \frac{1}{z} \end{aligned}$$

Similar to the lump-sum case, money in this system is neutral but not super-neutral since z changes μ (money growth affects consumption smoothing). Suppose the agent's utility function is $ln(c_{t,1}) + \beta ln(c_{t,2})$, then we know that the equilibrium condition is:

$$\frac{e_2 + s_{t+1}/z}{\beta(e_1 - s_t)} = \frac{s_{t+1}}{s_t z}$$

In steady-state, we have $s_t = s_{t+1} = s^*$ and thus

$$\frac{e_2 + s^*/z}{\beta(e_1 - s^*)} = \frac{s^*}{s^* z} \ \Rightarrow \ s^* = \frac{\beta e_1 - z e_2}{1 + \beta}$$

The government must balance its budget, so

$$\begin{aligned} g_t &= q_t(M_{t+1} - M_t) = q_t(z M_t - M_t) = (z-1) M_t q_t = (z-1) s_t \\ \Rightarrow \ g^* &= (z-1) s^* = (z-1) \frac{\beta e_1 - z e_2}{1 + \beta} \end{aligned}$$

The welfare-maximizing z is thus characterized by:

$$\frac{\partial}{\partial z}g^* = \frac{\beta e_1 - z e_2}{1+\beta} - (z-1)\frac{e_2}{1+\beta} = 0 \;\Rightarrow\; z^* = \frac{\beta e_1 + e_2}{2e_2}$$

8.5 OLG with Production

Now consider a slightly different environment than the basic model:

- $t = 1, 2, 3, \ldots$
- Each generation works when they are young, and consumes returns of investments on capital when they are old.
- The economy has a Constant-Returns-to-Scale production function

$$Y_t = F(K_t, N_t),\; y_t = f(k_t)$$

- $\frac{N_{t+1}}{N_t} = 1 + n$
- Agents' utility function is time-separable: $u(c_{t,1}, c_{t,2}) = u(c_{t,1}) + \beta u(c_{t,2})$

8.5.1 Competitive Equilibrium

The representative agent's problem is

$$\max\; u(c_{t,1}, c_{t,2}) \; s.t. \; \begin{cases} c_{t,1} = w_t - s_t \\ c_{t,2} = (1 + r_{t+1})s_t \end{cases}$$

The F.O.C. is

$$u'(c_{t,1}) = \beta(1 + r_{t+1})u'(c_{t,2}) \;\Rightarrow\; s_t = s(w_t, r_{t+1})$$

So the savings rate depends on w_t and r_{t+1} (makes intuitive sense). The firms' problem is:

$$\max_{N_t, K_t}\; F(K_t, N_t) - w_t N_t - r_t K_t$$

The F.O.C.s are thus:

$$[N_t] : w_t = F_N(K_t, N_t) = \frac{\partial}{\partial N_t}[N_t \cdot f(k_t)] = f(k_t) + N_t f'(k_t)\left(-\frac{K_t}{N_T^2}\right) = f(k_t) - f'(k_t)k_t$$

$$[K_t] : r_t = F_K(K_t, N_t) = \frac{\partial}{\partial K_t}[N_t \cdot f(k_t)] = N_t f'(k_t)\frac{1}{N_t} = f'(k_t)$$

Suppose that the firms' production function is Cobb-Douglas that accounts for depreciation so that $Y_t = K_t^\alpha N_t^{1-\alpha} - \delta K_t$, $y_t = k_t^\alpha - \delta k_t = f(k_t)$, then we have

$$\begin{aligned} w_t &= f(k_t) - f'(k_t)k_t = k_t^\alpha - \delta k_t - \alpha k_t^{\alpha-1} \cdot k_t - \delta k_t = (1-\alpha)k_t^\alpha \\ r_t &= f'(k_t) = \alpha k_t^{\alpha-1} - \delta \end{aligned}$$

Then we have the market-clearing conditions:

- Capital Market: $\underbrace{K_{t+1} - K_t}_{\substack{\text{Investments} \\ \text{by } G_t}} = \underbrace{N_t S_t - K_t}_{\substack{\text{Net Savings} \\ \text{in the Economy}}} \Rightarrow K_{t+1} = N_t s_t$

  This gives us the low of motion of capital as:

$$s_t = \frac{K_{t+1}}{N_t} = \frac{K_{t+1}}{N_{t+1}}\frac{N_{t+1}}{N_t} = (1+n)k_{t+1} \Rightarrow s\left(f(k_t) - f'(k_t)k_t, f'(k_{t+1}\right) = (1+n)k_{t+1} \quad (\odot)$$

- Labor market: $w_t = f(k_t) - f'(k_t)k_t$, but this is already accounted for when we want the capital market to clear. As such, generally we think about the labor supply to be simply N_t

To understand the relationship between savings rate and equilibrium path for capital, we can differentiate equation $(\odot)$ with respect to k_t and get:

$$\begin{aligned} (1+n)\frac{\partial}{\partial k_t}k_{t+1} &= s_w[f'(k_t) - f''(k_t)k_t - f'(k_t)] + s_r\left[f''(k_{t+1})\frac{\partial k_{t+1}}{\partial k_t}\right] \\ \Rightarrow \frac{\partial k_{t+1}}{\partial k_t} &= \frac{-s_w f''(k_t)k_t}{1+n+s_r f''(k_{t+1})} \quad (\star) \end{aligned}$$

To determine the sign of s_w, we can use the RA's problem:

$$u'(w_t - s_t) = \beta(1+r_{t+1})u'((1+r_{t+1})s_t)$$

Differentiating both sides w.r.t w_t we get:

$$u''(w_t - s_t)[1 - s_w] = \beta(1+r_{t+1})u''((1+r_{t+1})s_t)(1+r_{t+1})s_w$$

Solving for s_w, we get (assuming $u'' < 0$)

$$s_w = \frac{u''(c_{t,1})}{u''(c_{t,1}) + \beta(1+r_{t+1})^2 u''(c_{t,2})} > 0$$

The sign of s_r is harder to determine, as the substitution effect ($s \uparrow$) and income effect ($s \downarrow$) work in opposite directions. Empiricallly, substitution effect dominates and $s_r > 0$. Plugging these into equation ($\star$), we get:

$$\frac{\partial k_{t+1}}{\partial k_t} = \frac{-s_w f''(k_t)k_t}{1+n+s_r f''(k_{t+1})} > 0$$

Example: Cobb-Douglas Production and Log Utility

Let $u(c) = ln(c)$ and $f(k) = k^\alpha - \delta k$.
Representative Agent's Problem:

$$\max_{s_t} \; ln(w_t - s_t) + \beta ln((1+r_{t+1})s_t)$$

The F.O.C. is

$$\frac{1}{w_t - s_t} = \beta \frac{1}{s_t} \;\Rightarrow\; s_t = s(w_t) = \frac{\beta}{1+\beta} w_t$$

In the case of log utility, the savings rate does not depend on r_{t+1}.
Firm's Problem:

$$\begin{aligned} w_t &= f(k_t) - f'(k_t)k_t = (1-\alpha)k_t^\alpha \\ r_t &= f'(k_t) = \alpha k_t^{\alpha-1} - \delta \end{aligned}$$

For markets to clear, we must have

$$(1+n)k_{t+1} = s\left((1-\alpha)k_t^\alpha\right) \;\Rightarrow\; k_{t+1} = \frac{\beta(1-\alpha)k_t^\alpha}{(1+\beta)(1+n)}$$

In steady-state, we have $k_t = k_{t+1} = k^*$ so

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+\beta)(1_n)}\right]^{\frac{1}{1-\alpha}}$$

8.5.2 Social Planner's Problem

The social planner's problem in this economy is:

$$\max_{\{c_{t,2},k_{t+1}\}} \sum_{t=0}^{\infty} \frac{1}{(1+R)^t}[u(c_{t,1}) + \beta u(c_{t,2})]$$
$$s.t.\ f(k_t) = (1+n)k_{t+1} - k_t + c_{t,1} + \frac{1}{1+n}c_{t-1,2}$$

where R is the **social planner's discount rate** for each generation's welfare[33]. For generation t, the F.O.C.s are:

$$[c_{t,2}] : \beta u'(c_{t,2}) = \frac{u'(c_{t+1,1})}{(1+R)(1+n)}$$
$$[k_{t+1}] : u'(c_{t,1}) = \frac{u'(c_{t+1,1})[f'(k_{t+1}) + 1]}{(1+R)(1+n)}$$

Combining these two, we get

$$u'(c_{t,1}) = \beta u'(c_{t,2})[f'(k_{t+1} + 1]$$

At a glance, this seems identical to the competitive equilibrium case:

$$[CE] : u'(c_{t,1}) = \beta u'(c_{t,2})(1 + r_{t+1}),\ r_{t+1} = f'(k_{t+1})$$
$$[SP] : u'(c_{t,1}) = \beta u'(c_{t,2})[f'(k_{t+1}) + 1]$$

But they are actually quite different, because R causes dynamic inefficiencies. See that, in steady-state, we have $c_{t,1} = c_{t+1,1} = c_1^*$. From the F.O.C. using k_{t+1}, we get that

$$u'(c_1^*) = \frac{u'(c_1^*)[f'(k_{t+1}) + 1)]}{(1+R)(1+n)} \ \Rightarrow\ f'(k_{t+1}) = (1+R)(1+n) - 1 = R + n + Rn$$

both R and n are really small, so $Rn \approx 0$. As such, the social planner steady-state capital, which we shall call **Modified Golden Rule** is $k^{MGR} = k_{sp}^*$ such that $f'(k_{sp}^*) \approx n + R$.

This is analogous to the Golden Rule Capital in the Neoclassical Growth model where the social planner wants to deplete resources every period so said period can have maximal consumption[34].

[33] This means that if $R > 0$, the social planner cares more about current generations than future generations

[34] There were some more detailed discussions in Younghee's original notes, but I only wanted to capture the gist here.

As such, dynamic inefficiency[35] can arise in the competitive equilibrium case through over-saving ($f' = r$ v.s. $f' = n + R$).

If $r < n+R$, then the competitive equilibrium steady-state is dynamically inefficient because the social planner can (and wants to) redistribute so that people can have higher consumption. Similarly, if $r > n + R$, the steady-state is already dynamically efficient because the social planner cannot redistribute resources to increase utility for every one.

8.6 OLG with Bequests/Social Security

Let R_b be the discount rate in the private sector[36], the agents in the private sector[37] have the Bellman Equation:

$$\begin{aligned}
V_t &= u(c_{t,1}) + \beta u(c_{t,2}) + \frac{1}{1+R_b} V_{t+1} \\
&= u(c_{t,1}) + \beta u(c_{t,2}) + \frac{1}{1+R_b} \left[u(c_{t+1,1}) + \beta u(c_{t+1,2})\right] + \left(\frac{1}{1+R_b}\right)^2 V_{t+2} \\
&\vdots \\
&= \sum_{s=t}^{\infty} \left[\frac{1}{1+R_b}\right]^{s-t} \left[u(c_{s,1}) + \beta u(c_{s,2})\right]
\end{aligned}$$

and the constraints are:

$$\begin{aligned}
c_{t,1} &= w_t - s_t + b_t \\
c_{t,2} &= (1 + r_{t+1}) s_t - b_{t+1}
\end{aligned}$$

where s_t is the agent's personal savings and b_t is transfers the agent receives from the previous generation. One way to study this is model is to think of bequests as social security.

8.6.1 Fully Funded Social Security: Save for your future

In period t, the agent's problem is

$$\max \ u(c_{t,1}) + \beta u(c_{t,2}) \ s.t. \ \begin{cases} c_{t,1} = w_t - s_t - d_t \\ c_{t,2} = (1 + r_{t+1})(s_t + d_t) \end{cases}$$

[35] Just remember that, whatever the social planner wants, is dynamically efficient.
[36] If $R_b = R$, then the competitive equilibrium is the same as the social planner's equilibrium
[37] Think of this as a family with generations after generations.

For the markets to clear, we must have

$$s_t + d_t = k_{t+1}(1+n) \;\Rightarrow\; d_t < k_{t+1}(1+n)$$

Since d_t is just another savings mechanism, it is actually redundant (unless the rates are better, which would violate the no-arbitrage condition and render s_t useless). As such, the fully-funded system will yield the same equilibrium result as the simple competitive equilibrium case.

8.6.2 Unfunded Social Security: Pay-As-You-Go

In period t, the agent's problem is

$$\max\; u(c_{t,1}) + \beta u(c_{t,2}) \; s.t. \; \begin{cases} c_{t,1} = w_t - s_t - d_t \\ c_{t,2} = (1+r_{t+1})s_t + \overbrace{(1+n)d_t}^{\substack{\text{Reaps the fruits of} \\ \text{young people's labor}}} \end{cases}$$

Assume that the contribution is constant over time ($d_t = d_{t+1} = d$), then we can solve the optimization problem:

$$u'(c_{t,1}) = \beta(1+r_{t+1})u'(c_{t,2}) \;\Rightarrow\; s_t = s(w_t, r_{t+1}, d)$$

Differentiate this equation with respect to d, we get

$$\begin{aligned} u''(c_{t,1})[-s_d - 1] &= \beta(1+r_{t+1})u''(c_{t,2})[(1+r_{t+1})s_d + (1+n)] \\ \Rightarrow\; s_d &= -\frac{u''(c_{t,1}) + \beta(1+r_{t+1}(1+n)u''(c_{t,2})}{u''(c_{t,1}) + \beta(1+r_{t+1})^2 u''(c_{t,2})} < 0 \end{aligned}$$

This means that when social security contribution increases, young people would save less. This makes intuitive sense as they still want to consume when young, and higher constant contribution means higher future payout, so savings is less necessary. In fact, if $|s_d| > 1$ (meaning $r < n$), people would save much less because the social security system gives much better "returns". If $|s_d| < 1$ (meaning $r > n$), then people would save less but not as outrageously less, as investment still yields better returns. The former will yield a lower steady-state capital stock than the latter.

9 Asset Pricing

9.1 Asset Prices in an Endowment Economy (Lucas Tree Model)

The environment in this model is:

- There are N trees in the economy that produce $y_t = (y_{1,t}, \dots, y_{N,t})$ fruits every period.
- The production of the trees follows a Markov process with CDF $F(y, y') = P(y_{t+1} < y' \mid y_t = y)$.
- There are N corresponding assets (denoted x_i) that entitles the owner of said asset the output of the corresponding tree in a given period.

The representative agent's problem is:

$$\max_{c_t} E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \; s.t. \; c_t + p_t x_t = \underbrace{(y_t + p_t)}_{\text{Dividends + Price}} \cdot x_{t-1}$$

Our goal is to figure out how p_t is determined.

Definition: An asset trade equilibrium is a list of $(p_t, V(y_t, x_{t-1}), x(y_t, x_{t-1})$ such that

- The representative agent maximizes utility
- $c_t = \sum_{i=1}^{n} y_{it}$
- $(x_t) = (1, 1, 1, \dots)$

Like any other equilibrium problem in this class, we want to set up our Bellman equation. In this case, we have state variables $\{y_t, z_{t-1}\}$ and the control variable $\{z_t\}$ (Note that in class it is possible that Andrei used z for z_{t-1} and x for z_t but I like to be consistent and intuitive.). Our Bellman equation is thus:

$$V(y_t, z_{t-1}) = \max_{z_t} \left\{ u\left(\underbrace{\sum_{j=1}^{\mathcal{J}} \overbrace{y_{j,t} z_{j,t-1}}^{\text{Dividends}} + \overbrace{p_{j,t}(z_{j,t-1} - z_{j,t})}^{\substack{\text{Price for Changes} \\ \text{in Asset Holding}}}}_{=c_t} \right) + \beta E\left[V(y_{t+1}, z_t)\right] \right\}$$

So our F.O.C.s <u>for each asset j</u> are:

$$[\partial z_{j,t}] : \; -u'(c_t) p_{j,t} + \beta E\left[V_{z_{i,t}}\right] = 0$$

$$[\partial z_{j,t-1}]: V_{z_{j,t)1}} = u'(c_t)[y_{j,t} + p_{j,t}] \;\Rightarrow\; V_{z_{j,t}} = u'(c_{t+1})[y_{j,t+1} + p_{j,t+1}]$$

Combining these two, we get the infamous **Asset Pricing Equation**:

$$u'(c_t)p_{j,t} = \beta E\left[u'(c_{t+1})(y_{j,t+1} + p_{j,t+1})\right]$$

Example:

Suppose that our representative agent is *risk-neutral*, so $u(c) = A \cdot c + B \;\Rightarrow\; u'(c) = A$. Using the *asset pricing equation* above, we get

$$\begin{aligned}
Ap_{j,t} &= \beta E\left[A(y_{j,t+1} + p_{j,t+1})\right] = \beta E\left[Ay_{j,t+1} + Ap_{j,t+1}\right] \\
p_{j,t} &= \beta E\left[y_{j,t+1} + \overbrace{\beta E\left[y_{j,t+2} + p_{j,t+2}\right]}^{=p_{j,t+1}}\right] \\
&= \beta E\left[y_{j,t+1} + \beta E\left[y_{j,t+2} + \beta E\left[y_{j,t+3} + Ap_{j,t+3}\right]\right]\right] \\
&= \beta E\left[y_{j,t+1} + \beta E\left[y_{j,t+2} + \beta E\left[y_{j,t+3} + \beta E\left[y_{j,t+4} + \cdots\right]\right]\right]\right] \\
&= E\left[\sum_{s=t+1}^{\infty} \beta^{s-t} y_{j,s}\right]
\end{aligned}$$

For a more general case (such as CRRA), we must assume a version of the *Transversality Condition*- $\lim_{t\to\infty} \beta^t p_{j,s+t} = 0$. With such assumption, we get

$$p_{j,t} = \beta E\left[\frac{u'(c_{t+1}}{u'(c_t)}[y_{j,t+1} + p_{j,t+1}]\right] = \cdots = E\left[\sum_{s=t+1}^{\infty} \beta^{s-t} \underbrace{\frac{u'(c_s)}{u'(c_t)}}_{\substack{\text{Subjective}\\ \text{Discount}\\ \text{Rate}}} y_{j,s}\right]$$

9.2 The Fundamental Price

When looking to price assets, one must wonder whether a steady-state price solution is possible. If so, then the in steady-state we must have

$$c_t = d_t = y_t$$

As prices stabilize, so do holdings, and agents would simply only consume dividends from the holdings. We posit that a sequence of prices (p_t^f) is on the path to a steady-state equilibrium, and call this the **Fundamental Solution/Price** that is decided by the stream of dividends (determined exogenously[38]). The fundamental solution thus must satisfy:

$$\begin{aligned} p_{j,t}^f u'(c_t) &= \beta E\left[u'(c_{t+1})\left(p_{j,t+1}^f + c_{t+1}\right)\right] \\ &= \beta E\left[u'(c_{t+1})c_{t+1}\left(\beta E\left[u'(c_{t+2})\left(p_{j,t+2}^f + c_{t+2}\right)\right]\right)\right] \\ &= \cdots \\ \Rightarrow\ p_{j,t}^f &= E\left[\sum_{s=t+1}^{\infty} \beta^{s-t}\frac{u'(c_s)}{u'(c_t)}c_s\right] \end{aligned} \tag{1}$$

However, we know that, empirically, we don't actually get stable price over time. Like when we studied growth and business cycle, we shall now posit that real prices are determined by the fundamental price as well as noise/bubbles[39]. So a more realistic price is:

$$p_{j,t} = p_{j,t}^f + B_{j,t}$$

Using our *asset pricing equation*, we then get:

$$\left(p_{j,t}^f + B_{j,t}\right)u'(c_t) = \beta E\left[u'(c_{t+1})\left(c_{t+1} + p_{j,t+1}^f + B_{j,t+1}\right)\right] \tag{2}$$

Subtract equation (1) from equation (2), we get the price changes due solely to bubbles:

$$B_{j,t}u'(c_t) = \beta E\left[u'(c_{t+1})B_{j,t+1}\right]$$

In steady-state, we have $c_t = c_{t+1} = \bar{c}$ so this equation in steady-state is:

$$B_{j,t} = \beta E\left[B_{j,t+1}\right] \ \Rightarrow\ E\left[B_{j,t+1}\right] = \frac{B_{j,t}}{\beta} \tag{3}$$

Equation (3) tells us that bubbles are values that stem purely from speculation of future bubbles. If current asset is under-priced, it means that the future price is expected to decrease.

[38] It is possible to introduce this in a production economy where the stream of dividends is endogenously determined. However, this makes the model much more complicated, and so we will not touch on that here.

[39] Notice that this means a risk-free asset should be priced at the fundamental price, since there is no speculation.

9.3 The Determinants of the Variability of Stock Prices (CAPM)

Using $c_t = d_t = y_t$ and the asset pricing equation $u'(c_t)p_{j,t} = \beta E\left[u'(c_{t+1})\left(c_{t+1} + p_{j,t}\right)\right]$, we get

$$1 = E\left[\underbrace{\frac{\beta u'(c_{t+1})}{u'(c_t)}}_{\text{Call this } S_t} \cdot \underbrace{\frac{p_{j,t+1} + c_{t+1}}{p_{j,t}}}_{\text{Call this } R_{j,t}}\right]$$

$$\Rightarrow \ 1 = E[S_t \cdot R_{j,t}] = Cov(S_t, R_{j,t}) + E[S_t]E[R_{j,t}]$$

$$\Rightarrow \ E[R_{j,t}] = \frac{1}{E[S_t]}\left[1 - Cov(S_t, R_{j,t})\right]$$

Intuitively, when MRS is high, that means either c_t is high or c_{t+1} is low (so when in periods where current consumption is needed and future consumption is sacrificed). If $R_{j,t}$ is positively correlated with MRS, it means asset j gives high returns during times when high current consumption is needed. Asset j then provides insurance to the agent, and hence the overall expected return in equilibrium for asset j is low.

On the other hand, if $R_{k,t}$ is negatively correlated with MRS, it means that asset k only gives high returns when current consumption needs to be low and future consumption is wanted (MRS is low). Asset k then is the opposite of insurance, and hence the overall expected return in equilibrium for asset k must be high.

9.4 Equity Premium Puzzle (Mehra-Prescott)

Using historical data, the theoretical stock market return $R^e - R^f$ (expected return minus risk-free returns) should be less than 1%. However, the actual return is closer to 6%. Mehra-Prescott proposes the following model:

Consider a growing economy with $y_{t+1} = x_t y_t$ where $x_t = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the stochastic gross growth rate following the Markovian transition matrix:

$$\Phi = \begin{pmatrix} \Phi_{11} & \cdots & \Phi_{1n} \\ \vdots & \ddots & \vdots \\ \Phi_{n1} & \cdots & \Phi_{nn} \end{pmatrix}, \quad \Phi_{ij} = P(x_{t+1} = \lambda_J \mid x_t = \lambda_i)$$

Note that this means $y_s = x_{s-1}y_{s-1} = x_{s-1}x_{s-2}y_{s-2} = \cdots = y_t \prod_{i=t}^{s-1} x_i$.

Let agents have be CRRA, so $u(c) = \frac{c^{1-\alpha}-1}{1-\alpha}$. Using the asset pricing equation, we get

$$\begin{aligned}
p_{j,t} &= E\left[\sum_{s=t+1}^{\infty} \beta^{s-t}\frac{u'(c_s)}{u'(c_t)}c_s\right] = E\left[\sum_{s=t+1}^{\infty} \beta^{s-t}\frac{c_s^{-\alpha}}{c_t^{-\alpha}}c_s\right] = E\left[\sum_{s=t+1}^{\infty} \beta^{s-t}\left(\frac{c_s}{c_t}\right)^{1-\alpha} c_t\right] \\
&= E\left[\sum_{s=t+1}^{\infty} \beta^{s-t}\left(\frac{y_s}{y_t}\right)^{1-\alpha} c_t\right] = E\left[\sum_{s=t+1}^{\infty} \beta^{s-t}\left(\prod_{i=t}^{s-1} x_i\right)^{1-\alpha} c_t\right] \\
&= c_t \cdot \underbrace{E\left[\sum_{s=t+1}^{\infty} \beta^{s-t}\left(\prod_{i=t}^{s-1} x_i\right)^{1-\alpha}\right]}_{\text{Call this } w_j} \\
\Rightarrow\ & p_{j,t} = w_j c_t
\end{aligned}$$

Denote the general transition for state a at period t and state b at period $t+1$, the asset pricing equilibrium must satisfy $p_{a,t} = w_a c_t$ and

$$p_{a,t} = \beta E\left[\frac{u'(c_{t+1})}{u'(c_t)}(c_{t+1} + p_{a,t+1})\right] = \beta E\left[\frac{c_{t+1}^{-\alpha}}{c_t^{-\alpha}}[w_b c_{t+1} + c_{t+1}]\right]$$

Combining these two equations, we get:

$$\begin{aligned}
w_a c_t &= \beta E\left[\frac{c_{t+1}^{-\alpha}}{c_t^{-\alpha}}[w_b c_{t+1} + c_{t+1}]\right] = \beta E\left[c_t^{\alpha} \cdot \underbrace{c_{t+1}^{1-\alpha}}_{=(\lambda_b c_t)^{1-\alpha})} \cdot (1+w_b)\right] \\
&= \beta E\left[(1+w_b)\lambda_b^{1-\alpha} c_t\right] \\
\Rightarrow\ w_a &= \beta E\left[(1+w_b)\lambda_b^{1-\alpha}\right] = \beta \sum_{i=1}^{n} \underbrace{\Phi_{ib}}_{\substack{\text{Probability of}\\ \text{Transitioning}\\ \text{from } i \text{ to } b}} (1+w_b)\lambda_b^{1-\alpha}
\end{aligned}$$

The rate of return transitioning from state a to b is:

$$r_{ab}^e = \frac{p_{b,t+1}^e + c_{t+1} - p_{a,t}^2}{p_{a,t}^e} = \frac{w_b \lambda_b c_t + \lambda_b c_t - w_a c_t}{w_a c_t} = \frac{(w_b+1)\lambda_b - w_a}{w_a}$$

and so the expected rate of return at from state a in period t is:

$$r_a^e = \sum_{b=1}^{n} \Phi_{ab} r_{ab}^e$$

Example: Two States

Let $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ be the vector of probability such that $\pi\Phi = \pi$. Suppose there are only two possible states of the world, so

$$\begin{pmatrix}\pi_1 & \pi_2\end{pmatrix}\begin{pmatrix}\Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22}\end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_2\end{pmatrix} \Rightarrow \pi_1 = \pi_2 = \frac{1}{2}$$

The return on equity is thus $R^e = \sum_{i=1}^{2} \pi_i r_i^e$. Similarly, we can calculate the return on risk-free assets (think of these as 1-period bonds for simplicity) as:

$$\begin{aligned}
p_{a,t}^f &= \beta E\left[\frac{c_t^\alpha}{c_{t+1}^\alpha}(\underbrace{1}_{y_{t+1}=1 \text{ risk-free}} + \underbrace{0}_{\text{1-period bond}})\right] = \beta E\left[\frac{c_t^\alpha}{\lambda_b^\alpha c_t^\alpha}\right] = \beta \sum_{b=1}^{n} \Phi_{ab}\lambda_b^{-\alpha} \\
\Rightarrow\ r_a^f &= \frac{p_{b,t+1}^f + c_{t+1} - p_{a,t}^f}{p_{a,t}^f} = \frac{0 + 1 - p_{a,t}^f}{p_{a,t}^f} = \frac{1}{p_{a,t}^f} - 1 \\
\Rightarrow\ R^f &= \sum_{i=1}^{2} \pi_i r_i^f
\end{aligned}$$

Empirically, we observed that

- Average growth rate of consumption is $g_c \approx 1.8\%$
- Standard deviation of consumption is 0.036
- $Corr(\lambda, \lambda') = -0.14$

So we can calculate Φ, R^e, R^f (but the result is far from what is actually observed).

In an attempt to explain the equity premium puzzle, we tried to see if whether the problem comes from risk attitude or preferences for consumption smoothing:

Attempt 1: Let α be the coefficient of Relative Risk Aversion:

Suppose that the average asset yields

$$\begin{cases} c + \alpha c & \text{with probability } \frac{1}{2} \\ c - \alpha c & \text{with probability } \frac{1}{2} \end{cases}$$

and so $u(c-\delta c) = \frac{1}{2}u(c-\alpha c) + \frac{1}{2}u(c+\alpha c)$ where $c - \delta c$ is the certainty equivalent. Using Taylor approximation, we get

$$
\begin{aligned}
u(c-\delta c) &\approx u(c) - \delta c u'(c) \\
RHS &\approx \frac{(\alpha c)^2 u''(c)}{2} \\
-\delta c u'(c) &= \frac{(\alpha c)^2 u''(c)}{2} \\
\Rightarrow \underbrace{\delta}_{\text{Risk Premium}} &= \underbrace{-\frac{u''(c)c}{u'(c)}}_{\text{Coefficient of Relative Risk Aversion}} \cdot \underbrace{\frac{\alpha^2}{2}}_{\text{Some Constant}}
\end{aligned}
$$

So a general form makes it difficult to figure out the risk premium.

Attempt 2: Let $\frac{1}{\alpha}$ be elasticity of intertemporal substitution

We know that the asset pricing equation suggests:

$$p_{j,t}u'(c_t) = \beta E\left[u'(c_{t+1})(p_{j,t+1} + y_{j,t+1})\right]$$

So for a 1-period risk free asset, we should get

$$
\begin{gathered}
R^f = \frac{u'(c_t)}{\beta E u'(c_{t+1})} = \frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^\alpha \Rightarrow \; ln(R^f) = \text{some constant} + \alpha ln(c_{t+1}) \\
\Rightarrow \text{ elasticity of intertemporal substitution is: } \frac{\partial ln(c_{t+1})}{\partial ln(R^f)} = \frac{1}{\alpha}
\end{gathered}
$$

Using the observed growth rate of consumption $g_c \approx 1.8\%$ and $\beta = 0.95$, in order to get $R^e - R^f \approx 6\%$, α needs to be 50, which is a ridiculously large elasticity of substitution.

People have proposed different ways to solve this puzzle. We won't go over them but you should know that they are

- Assume specific utility functions that separates risk aversion coefficients and elasticity of substitution
- Introduce some small probability of catastrophe
- Introduce weighted average of consumption in the past (habit formation) $u(c_t - \bar{c})$ so that consumption smoothing has a "set level" agents need to match.

10 Bargaining Theory

In studying market behaviors, we often need to split our analysis into **Extensive Margins** and **Intensive Margins**[40]. What we have studied thus far separate these analyses, but the studies of bargaining theory tries to bring them together. In general, we can take either the **Axiomatic Approach** pioneered by John Nash, or the **Strategic Approach** formalized by Ariel Rubenstein. We shall briefly discuss both of these, and introduce some simple models:

10.1 The Axiomatic Approach

Consider the following game:

- Let $\mathcal{I} = \{1, 2\}$ be the set of 2 players
- Let u_i be the payoffs of the player
- Let A be the set of agreements/allocations between the two players
- Let D be the unique disagreement/breakdown outcome
- Let $S = \{(u_1(a), u_2(a)) \mid a \in A\}$ be the set of payoffs given allocation
- Let $d_i = u_i(D)$ be the "threats" that a player can make (think Nash reversion)

A bargaining game is (S, D) such that the solution to the game, $f(s, d)$, satisfies the following 4 axioms:

Axiom 1: Invariance of Equivalent Utility Representation

Take (S, D) and (S', D') where $S_i' = \alpha_i S_i + \beta_i$, $d_i' = \alpha_i d_i + \beta_i$, then we must have

$$f_i(s', d') = \alpha_i f_i(s, d) + \beta_i$$

Axiom 2: Symmetry If $d_1 = d_2$ and $(s_1, s_2) \in S$, then $(s_2, s_1) \in S$ and $f_1(s, d) = f_2(s, d)$

Axiom 3: Pareto Efficiency If $\exists s' \in S$ such that $s_i' > s_i$ and $s_j' \geq s_j$, then $s \neq f(s, d)$

[40]Extensive margins problems are about studying participation in the market (like long-run/ex-ante analyses). This framework helps us understand what leads to entry/exit, which change the environment of the market in equilibrium. Intensive margins, on the other hand, is the study of allocations (like short-run/ex-post analyses). If an agent decides to be in the market, how do they determine where to allocate their resources in the market. For example, in studying the labor market, we often think about labor force participation as an extensive margin problem, and wage distribution as an intensive margin problem.

Axiom 4: Independence of Irrelevant Alternatives

Take (S, D) and (S', D) where $S \subset S'$. If $f(S', d) \in S$ then $f(s, d) = f(s', d)$. You can think of this like condition α in Jon Eguia lectures.

Theorem 10.1: Symmetric Nash Bargaining

Take the game (S, D) where S is convex. Then there exists a unique solution $f(s, d)$ that satisfies all axioms and

$$f(s, d) = \underset{s \in S}{argmax}\ (s_1 - d_1)(s_2 - d_2)$$

such that $s_i > d_i$.
If we relax Axiom (2), then we can write

$$f(s, d) = \underset{s \in S}{argmax}\ (s_1 - d_1)^{\theta}(s_2 - d_2)^{1-\theta}$$

where $\theta \in [0, 1]$ represents bargaining power of player 1.

Example: Risk-Neutral Nash Bargaining

Consider the following Nash Bargaining Problem:

$$\max_q [u_1(q) - d_1]^{\theta}[u_2(q) - d_2]^{1-\theta}$$

The F.O.C. is

$$\begin{aligned} [q]:\ \theta[u_1(q) - d_1]^{\theta_1}[u_2(q) - d_2]^{1-\theta}u_1' + (1-\theta)[u_1(q) - d_1]^{\theta}[u_2(q) - d_2]^{-\theta}u_2' = 0 \\ \theta(u_2 - d_2)u_1' + (1-\theta)(u_1 - d_1)u_2' = 0 \end{aligned}$$

Let $u_1(q) = 1$ and $u_2(q) = 1 - q$, this means the F.O.C. is

$$\begin{aligned} &\theta(1 - q - d_2) = (1-\theta)(q - d_1) \\ \Rightarrow &\begin{cases} q = d_1 + \theta(1 - d_1 - d_2) \\ 1 - q = d_2 + (1-\theta)(1 - d_1 - d_2) \end{cases} \\ \Rightarrow &\text{ NE Division of Surplus for Risk Neutral Agents} \end{aligned}$$

Example: Worker-Firm Nash Bargaining

Consider the following Nash Bargaining Problem:

$$\max_q [u(c, 1-h) - d_1]^{\theta} [F(h) - c - d_2]^{1-\theta}$$

The F.O.C. is

$$[c]: \; \theta(F(h) - c - d_2)u_c + (1-\theta)(u(c, 1-h) - d_1)(-1) = 0$$
$$[h]: \; \theta(F(h) - c - d_2)u_h + (1-\theta)(u(c, 1-h) - d_1)F_h = 0$$

Dividing the first F.O.C. by the second we get

$$MRS = \frac{u_c}{u_h} = \frac{1}{F_h} = MRT$$

10.2 The Strategic Approach

Consider the following environment:

- Infinite Horizon Discrete Time: $t = 0, 1, 2, \ldots$
- 2 players bargain to split the total surplus of 1
- In each period, one player makes an offer and the other decides whether to accept or reject. If it is accepted, the game ends. If it is rejected, we move to the next period.
- Let q be the share of surplus that goes to player 1
- Let $u_1(q)$ and $u_2(q)$ be the players' utility if q share goes to player 1
- There is a common discount rate δ

Result: The equilibrium is characterized by a set of reservation shares (q_1, q_2), the subscripts denote the player that made the offer.

SPNE: Indifference Conditions
Using backwards induction,

P_2 offers

$$\underbrace{q_2}_{\substack{P_1 \text{ gets } q_2 \text{ if she} \\ \text{accepts the offer}}} = \underbrace{\delta q_1}_{\substack{P_1 \text{ gets } \delta q_1 \text{ if } P_2 \\ \text{accepts } P_1\text{'s offer} \\ \text{next period}}}$$

and P_1 offers

$$\underbrace{1 - q_1}_{\substack{P_2 \text{ gets } 1-q_1 \text{ if she} \\ \text{accepts the offer}}} = \underbrace{\delta(1 - q_2)}_{\substack{P_2 \text{ gets } \delta(1-q_2) \text{ if } P_1 \\ \text{accepts } P_2\text{'s offer} \\ \text{next period}}}$$

Combining these two equations, we get (at $t = 0$):

$$q_1 = \frac{1}{1+\delta},\ q_2 = \frac{\delta}{1+\delta}$$

Here's a more generalized version:

Let M be the largest q that player 1 will get in any subgame. By backward induction,

At $t = 2$ (even period $\Rightarrow$ P_1 makes the offer): P_1 offers to get M and give P_2 $(1 - M)$

At $t = 1$ (odd period $\Rightarrow$ P_2 makes the offer): P_2 offers to get at most $(1 - \delta M)$ and so P_1 gets at least (δM) (indifference for next period)

At $t = 0$ (even period $\Rightarrow$ P_1 makes the offer): P_1 offers to get $(1 - \delta(1 - \delta M))$ and so P_2 gets at least $(\delta(1 - \delta M))$

By construction, at $t = 0$, P_1 wants to receive at most M, so $M = 1 - \delta(1 - \delta M)$, doing some algebra we get that $M = \frac{1-\delta}{1-\delta^2} = \frac{1}{1+\delta}$.

Similarly, let m be the smallest q that player 1 will get in any subgame. By backward induction,

At $t = 2$ (even period $\Rightarrow$ P_1 makes the offer): P_1 offers to get at least m and give P_2 $(1-m)$

At $t = 1$ (odd period $\Rightarrow$ P_2 makes the offer): P_2 offers to get at least $\delta(1 - m)$ and so P_1 gets at most $1 - \delta(1 - m)$ (indifference for next period)

At $t = 0$ (even period $\Rightarrow$ P_1 makes the offer): P_1 offers to get at least $\delta(1-\delta(1-m))$ and so P_2 gets at most $1-\delta(1-\delta(1-m))$

By construction, at $t = 0$, P_1 wants to receive at least m, so $m = \delta(1-\delta(1-m))$, doing some algebra we get that $m = \frac{\delta}{1+\delta}$.

See that $m + M = \frac{\delta}{1+\delta} + \frac{1}{1+\delta} = 1$ and $m < M$, so different bargaining power (in this case, it's about who offers at $t = 0$) allows us to find different solutions (q_1, q_2) such that $q_1 \in [m, M]$, $q_2 = 1 - q_1$.

Approaching Continuous Time:
Now, consider each period to be of length Δ and let π_i be the probability that player i gets to make the offer in period t (so instead of going back and forth, there is a chance of a player making two consecutive offers), and the discount rate is $\delta_i = \frac{1}{1+r_i\Delta}$.

Using backwards induction, we get the **indifference conditions** for the players as:

$$u_i(q_i) = \frac{1}{1+r_i\Delta}\Bigg[\underbrace{\pi_i u_i(q_j)}_{\substack{\text{Player } i \text{ needs} \\ \text{to offer player } j \\ j\text{'s reservation}}} + \underbrace{\pi_j u_i(q_i)}_{\substack{\text{Player } j \text{ needs} \\ \text{to offer player } i \\ i\text{'s reservation}}}\Bigg]$$

For risk-neutral agents, this means:

$$u_1(q) = q;\ \ u_2(q) = 1 - q$$

Plugging this into the indifference conditions we get:

$$\begin{aligned} q_1 &= \frac{1}{1+r_1\Delta}(\pi_1 q_2 + \pi_2 q_1) \\ 1 - q_2 &= \frac{1}{1+r_2\Delta}[\pi_1(1-q_2) + \pi_2(1-q_1)] \end{aligned}$$

Doing some algebra, we can show that:

$$\begin{aligned} q_1 &= \frac{\pi_1 r_2}{r_1\pi_1 + r_2\pi_2 + r_1 r_2\Delta} \\ q_2 &= \frac{\pi_2 r_1 + r_1 r_2\Delta}{r_1\pi_1 + r_2\pi_2 + r_1 r_2\Delta} \end{aligned}$$

As $\Delta \to 0$, $q_1 = q_2$ and the average payoff is $q = \pi_1 q_1 + \pi_2 q_2$. Using Taylor approximation, we can rewrite $u_1(q)$ and $u_2(q)$ as:

$$\begin{aligned}
u_1(q_1) \approx u_1(q) + (q_1 - q)u_1'(q) &\approx \frac{1}{1+r_1\Delta}\left\{\pi_1\left[u_1(q) + (q_2 - q)u_1'(q)\right] + \pi_2\left[u_1(q) + (q_1 - q)u_1'(q)\right]\right\} \\
&\approx \frac{1}{1+r_1\Delta}\left[\pi u_1 + \pi(q_2 - q)u_1' + \pi_2 u_1 + \pi_2(q_1 - q)u_1'(q)\right] \\
&= \frac{1}{1+r\Delta}\left[\underbrace{(\pi_1 + \pi_2)}_{=1} u_1 + \underbrace{(\pi_1 q_2 + \pi_2 q_2)}_{=q_2} u_1' - \underbrace{(\pi_1 + \pi_2)}_{=1} q u'\right] \\
&= \frac{1}{1+r_1\Delta} u_1(1) \\
u_1(q) + (q_1 - q)u_1'(q) &\approx \frac{1}{1+r_1\Delta} u_1(q)
\end{aligned}$$

Similarly,

$$\begin{aligned}
u_2(q_2) \approx u_2(q) + (q_2 - q)u_2'(q) &\approx \frac{1}{1+r_2\Delta}\left\{\pi_1\left[u_2(q) + (q_2 - q)u_2'(q)\right] + \pi_2\left[u_2(q) + (q_1 - q)u_2'(q)\right]\right\} \\
&= \frac{1}{1+r_2\Delta} u_2(q) \\
u_2(q) + (q_2 - q)u_2'(q) &\approx \frac{1}{1+r_2\Delta} u_2(q)
\end{aligned}$$

Multiply $u(q_1)$ by $(1+r_1\Delta)(1+r_2\Delta)\pi_2 u_2'$ and $u(q_2)$ by $(1+r_1\Delta)(1+r_2\Delta)\pi_1 u_1'$ and we get:

$$\begin{aligned}
(1+r_1\Delta)(1+r_2\Delta)\pi_2 u_2' u_1 + (1+r_1\Delta)(1+r_2\Delta)\pi_2 u_2'(q_1 - q)u_1' &= (1+r_2\Delta)\pi_2 u_2' u_1 \\
(1+r_1\Delta)(1+r_2\Delta)\pi_1 u_1' u_2 + (1+r_1\Delta)(1+r_2\Delta)\pi_1 u_1'(q_2 - q)u_2' &= (1+r_1\Delta)\pi_1 u_1' u_2
\end{aligned}$$

Adding these two equations together (notice $q_1 + q_2 = 1 = 2q$), we get

$$\begin{aligned}
(1+r_1\Delta)(1+r_2\Delta)\pi_2 u_1 u_2' + (1+r_1\Delta)(1+r_2\Delta)\pi_1 u_1' u_2 &= (1+r_2\Delta)\pi_2 u_1 u_2' + (1+r_1\Delta)\pi_1 u_1' u_2 \\
\Rightarrow\ r_1\Delta(1+r_2\Delta)\pi_2 u_1 u_2' + r_2\Delta(1+r_1\Delta)\pi_1 u_1' u_2 &= 0 \\
\Rightarrow\ r_1(1+r_2\Delta)\pi_2 u_1 u_2' + r_2(1+r_1\Delta)\pi_1 u_1' u_2 &= 0
\end{aligned}$$

So as $\Delta \to 0$, we get

$$r_1 \pi_2 u_1 u_2' + r_2 \pi_1 u_1' u_2 = 0$$

which is the solution to the strategic bargaining problem.

The Nash Bargaining problem (with threat points 0) here is:

$$\max_q \; [u_1 - d_1]^\theta [u_2 - d_2]^{1-\theta} = u_1(q)^\theta u_2(q)^{1-\theta}$$

The F.O.C. (plugging in the solution from the strategic bargaining) is:

$$[\partial q]: \;\; \theta u_1^{\theta-1} u_2^{1-\theta} u_1' + (1-\theta) u_1^\theta u_2^{-\theta} u_2' = 0 \;\Rightarrow\; \theta = \frac{r_2 \pi_1}{r_1 \pi_2 + r_2 \pi_1}$$

So if the bargaining power θ is

$$\theta = \frac{r_2 \pi_1}{r_1 \pi_2 + r_2 \pi_1}$$

then the Nash Bargaining problem is equivalent to the Strategic Bargaining problem.

Introducing Exogenous Breakdowns

To make the strategic bargaining problem more closely related to the Nash Bargaining problem, we can add in a exogenous probability for breakdowns (like threat points). Consider the following modifications:

- Let $\lambda_i \cdot \Delta$ be the exogenous probability of a breakdown
- Let λ_i follow a Poisson process
- Let b_i be player i's payoff in case of a breakdown

The **indifference conditions** then need to be rewritten as:

$$u_1(q_1) = \frac{1}{1 + r_1 \Delta} \left\{ \lambda_1 \Delta b_1 + (1 - \lambda_1 \Delta) \left[\pi_1 u_1(q_2) + \pi_2 u_1(q_1) \right] \right\}$$
$$u_2(q_2) = \frac{1}{1 + r_2 \Delta} \left\{ \lambda_2 \Delta b_1 + (1 - \lambda_2 \Delta) \left[\pi_1 u_2(q_2) + \pi_2 u_2(q_1) \right] \right\}$$

Using Taylor approximation around the average payoff q, we get

$$\frac{\pi_2}{r_2 + \lambda_2} \left[u_1 - \frac{\lambda_1 b_1}{r_1 + \lambda_1} \right] u_2' + \frac{\pi_1}{r_1 + \lambda_1} \left[u_2 - \frac{\lambda_2 b_2}{r_2 + \lambda_2} \right] u_1' = 0$$

So we can establish the sufficient conditions for equivalence of the two approaches:

$$\begin{cases} \theta = \frac{\pi_1(r_2+\lambda_2)}{\pi_1(r_2+\lambda_2)+\pi_2(r_1+\lambda_1)} \\ d_i = \frac{\lambda_i b_i}{r_i+\lambda_i} \end{cases} \Rightarrow GNBP \equiv SBP$$

Remark: For the rest of this entire section, please always recall that any strategy profiles constructed through backwards induction is an SPNE. This is important because that means we can solve the extensive margin problem separately from the intensive margin problem by using a placeholder "assumed" intensive margin solution (such as bargained wages, prices, etc.). The *Trejos and Wright* model is a good example of this.

10.3 Search, Bargaining, Money, and Prices (Trejos and Wright, JPE 1998)

Consider a similar environment as the one in the Kiyotaki & Wrigth model:

- Infinite Horizon Discrete Time: $t = 0, 1, 2, \ldots$
- Let β be the probability of meeting another agent (β follows a Poisson process)
- Let M be the fraction of people with fiat money in the economy
- Let $1-M$ be the fraction of people with goods to sell (no bartering and no consumption of own good)
- Let x be the single coincidence of buyer wanting a good from seller
- Let $u(q)$ and $c(q)$ denote the standard utility and cost functions
- When two a buyer meets a seller who is selling things the buyer wants (with probability βx), the bargaining begins

In this model, Trejos and Wright proposes that the trade process happens in 2 stages. Stage 1 is the extensive margin where agents need to decide whether to try to trade or not, and stage 2 is the intensive margin where agents bargain to find the "best price" for the good. For the extensive margins, they used a **Monetary Theory** approach, and for the intensive margins, they used a **Price Theory** approach.

10.3.1 Monetary Theory (Extensive Margin)

- Let Q be the exogenous quantity in the market. Assume that $\beta x = 1$.
- Let $V_{s,t}$ be the value of being a seller at the end of period t
- Let $V_{b,t}$ be the value of being a seller at the end of period t

We can write out the value functions as:

$$V_{s,t} = \frac{1}{1+r\Delta}\left\{ \underbrace{\beta x}_{=1} \cdot \overbrace{\Delta}^{\substack{\text{Period has}\\ \text{length } \Delta}} \cdot \underbrace{M}_{\substack{\text{Probability that}\\ \text{this seller}\\ \text{meets a buyer}}} \cdot \max\left\{ V_{b,t+\Delta} - c(Q), V_{s,t+\Delta} \right\} + \underbrace{(1-\Delta M)}_{\substack{\text{Probability of not}\\ \text{meeting a buyer}\\ \text{who wants to buy}}} V_{s,t+\Delta} \right\}$$

$$V_{b,t} = \frac{1}{1+r\Delta}\left\{ \Delta(1-M)\max\left\{ u(Q) + V_{s,t+\Delta}, V_{b,t+\Delta} \right\} + [1-\Delta(1-M)]V_{b,t+\Delta} \right\}$$

Multiply both equations by $1 + r\Delta$ and then subtract $V_{s,t}$ from the first equation we get:

$$r\Delta V_{s,t} = \Delta M \max\left\{ V_{b,t+\Delta} - c(Q), V_{s,t+\Delta} \right\} - \Delta M V_{s,t+\Delta} + V_{s,t+\Delta} - V_{s,t}$$

Dividing by Δ, we get:

$$rV_{s,t} = M \max\left\{ V_{b,t+\Delta} - c(Q) - V_{s,t+\Delta}, 0 \right\} + \frac{V_{s,t+\Delta} - V_{s,t}}{\Delta}$$

As $\Delta \to 0$, we get

$$rV_{s,t} = M \max\left\{ V_{b,t+\Delta} - c(Q) - V_{s,t+\Delta}, 0 \right\} + \dot{V}_s$$

Using the same process on $V_{b,t}$, we get

$$rV_{b,t} = (1-M)\max\left\{ u(Q) + V_{s,t+\Delta} - V_{b,t+\Delta}, 0 \right\} + \dot{V}_b$$

Define the functions $\varphi(Q)$ and $\psi(Q)$ as:

$$\varphi(Q) = -c(Q) + V_{b,t+\Delta} - V_{s,t+\Delta}$$
$$\psi(Q) = U(Q) - (V_{b,t+\Delta} - V_{s,t+\Delta})$$

In steady-state, we must have that $\dot{V}_s = \dot{V}_b = 0$ and $V_{i,t} = V_{i,t+\Delta}$. In a monetary equilibrium, we must also have $\varphi(Q) \geq 0$ and $\psi(Q) \geq 0$, so we get

$$\begin{aligned} rV_b - rV_s &= (1-M)u(Q) - (1-M)(V_{b,t} - V_{s,t}) + Mc(Q) - M(V_{b,t} - V_{s,t}) \\ \Rightarrow \; V_b - V_s &= \frac{(1-M)u(Q) + Mc(Q)}{1+r} \end{aligned}$$

Plugging this back into rV_s and rV_b and do some algebra, we get:

$$\begin{aligned} V_s &= \frac{M}{r}\left[\frac{(1-M)u(Q) + Mc(Q) - (1+r)c(Q)}{1+r}\right] \\ V_b &= \frac{1-M}{r}\left[\frac{-(1-M)u(Q) - Mc(Q) + (1+r)u(Q)}{1+r}\right] \end{aligned}$$

10.3.2 Price Theory (Intensive Margin)

Using the strategic approach, we define $\pi = \frac{1}{2}$ as the probability that a seller gets to be the first one to propose the price. Let q_s, q_b be the reservation *qualities* of seller and buyer. The notation is that q_s is the Buyer's reservation quality that the seller proposes. Let V_s and V_b be taken as given/solved in the extensive margins.

The Buyer's indifference condition is thus

$$\underbrace{u(q_s) + V_s}_{\substack{\text{Buyer's utility} \\ \text{plus continuation} \\ \text{value if buyer} \\ \text{accepts trade}}} = \underbrace{\frac{1}{1+r\Delta}}_{\substack{\text{Discount for future} \\ \text{value after rejecting} \\ \text{Seller's proposal } q_s}} \left\{ \underbrace{\frac{1}{2}}_{\substack{\text{Probability that} \\ \text{Buyer proposes}}} \cdot \underbrace{[u(q_b) + V_s]}_{\substack{\text{Value if Buyer} \\ \text{proposal is accepted}}} + \underbrace{\frac{1}{2}}_{\substack{\text{Probability that} \\ \text{Seller proposes}}} \cdot \underbrace{[u(q_s) + V_s]}_{\substack{\text{Value if Seller} \\ \text{proposal is accepted}}} \right\}$$

and the Seller's indifference condition is:

$$-c(q_b) + V_b = \frac{1}{1+r\Delta}\left\{\frac{1}{2}[-c(q_s) + V_b] + \frac{1}{2}[-c(q_b) + V_b]\right\}$$

Multiply the two equations by $(1 + r\Delta)$ and subtract themselves, we get:

$$\begin{aligned} r\Delta[u(q_s) + V_s] &= \frac{1}{2}[u(q_b) - u(q_s)] \\ r\Delta[-c(q_s b + V_b] &= \frac{1}{2}[c(q_b) - c(q_s)] \end{aligned}$$

Taking the ratio of the two, we get (recall that, as shown here, $\Delta \to 0 \;\Rightarrow\; q_s, q_b \to q$)

$$\frac{u(q_s) + V_s}{-c(q_b) + V_b} = \frac{u(q_b) - u(q_s)}{c(q_b) - c(q_s)} = \frac{\frac{u(q_b)-u(q_s)}{q_b - q_s}}{\frac{c(q_b)-c(q_s)}{q_b - q_s}} = \frac{u'(q)}{c'(q)}$$

If we use the Nash Bargaining Approach, we get (with threat points being 0):

$$\max\,[u(q)+V_s]\,[-c(q)+V_b]$$

with the F.O.C.

$$u'(q)[-c(q)+V_b]-c'(q)[u(q)+V_s]=0 \;\Rightarrow\; \frac{u(q)+V_s}{-c(q)+V_b}=\frac{u'(q)}{c'(q)}$$

So we can now formally describe the equilibrium in this model.

Definition: A Steady-State Search & Bargain Equilibrium is a list (V_s, V_b, Q) such that

- Agents maximize their utility by solving V_s, V_b (on the extensive margins)
- $q = Q$ such that (on the intensive margins)

$$\frac{u(Q)+V_s}{-c(Q)+V_b}=\frac{u'(Q)}{c'(Q)}$$

From the monetary equilibrium, we have

$$V_s=\frac{M}{r}\left[\frac{(1-M)u(Q)+Mc(Q)-(1+r)c(Q)}{1+r}\right]$$
$$V_b=\frac{1-M}{r}\left[\frac{-(1-M)u(Q)-Mc(Q)+(1+r)u(Q)}{1+r}\right]$$

Subbing these into the intensive margins problem we get

$$\underbrace{u'(q)\left[-c(q)+\frac{1-M}{r(1+r)}\left[(r+M)u(q)-Mc(q)\right]\right]-c'(q)\left[u(q)+\frac{M}{r(1+r)}\left[(1-M)u(q)-(1-M+r)c(q)\right]\right]}_{\text{We will call this function } T(q) \text{ such that we are in equilibrium when } T(q)=0}=0$$

From the solutions to the extensive margins problem, we know that

$$\varphi(q)=-c(q)+(V_b-V_s)=-c(q)+\frac{(1-M)u(q)+Mc(q)}{1+r}=\frac{(1-M)u(q)-(1-M+r)c(q)}{1+r}$$
$$\psi(q)=u(q)-(V_b-V_s)=u(q)-\frac{(1-M)u(q)+Mc(q)}{1+r}=\frac{(r+M)u(q)-Mc(q)}{1+r}$$

so we get

$$\varphi(q)\geq 0 \iff (1-M)[u(q)-c(q)]\geq rc(q) \;\Rightarrow\; u(q)>c(q)$$
$$\psi(q)\geq 0 \iff M[u(q)+c(q)]+ru(q)\geq 0$$

Recall that the Nash Bargaining Problem is:

$$q^E = \underset{q'}{argmax}\ [u(q') + V_s]\,[-c(q') + V_b]\ \ s.t.\ \begin{cases} -c(q) + V_b \geq V_s \\ u(q) + V_s \geq V_b \end{cases}$$

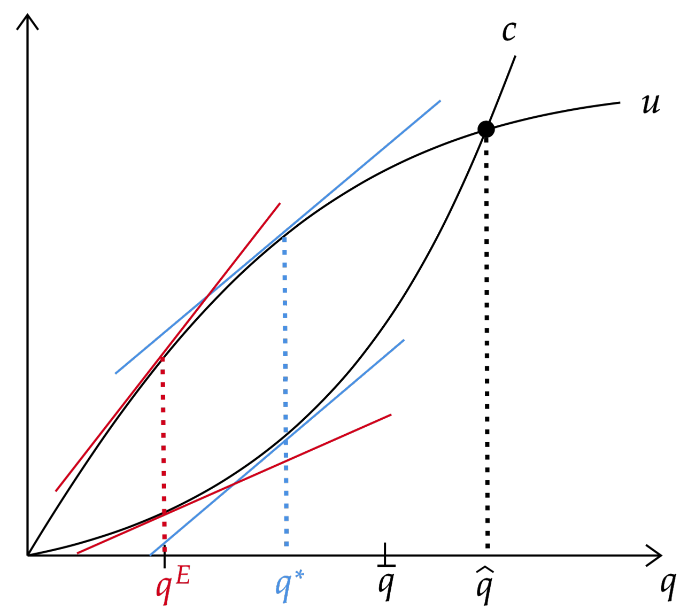
So thus far, we can actually define 4 different qualities:

- The optimal choice for the buyer q^* such that $u'(q^*) = c'(q^*)$
- The monopolistic (no consumer surplus) quality $\widehat{q}$ such that $u(\widehat{q}) = c(\widehat{q})$
- The knife-edge equilibrium quality $\bar{q}$ for the seller such that $\varphi(\bar{q}) = 0$ (Note that if $\varphi(\bar{q}) = 0$, then necessarily we have $T(\bar{q}) < 0$)
- The steady-state equilibrium quality q^E of the system such that $T(q^E) = 0$

We can properly qualify these solutions using $T(q)$. Notice that $T(q)$ is a function such that

- $T(0) = 0$
- $T'(0) > 0$ (given that $u'(0) < \infty$
- $T''(q) < 0$

The general result is that, if $r > 0$, then there exists a unique steady-state monetary equilibrium such that $u'(q^E) > c'(q^E)$. See the graph on the right for more clarity:
At q^E, $u'(q^E) > c'(q^E)$
At q^*, $u'(q^*) = c'(q^*)$
At $\bar{q}$, $\varphi(q) = 0 \ \Rightarrow\ T(q) < 0 \ \Rightarrow\ \bar{q} > q^E$



As in the monetary models studied in the past, we can measure social welfare with:

$$\begin{aligned} W = MV_b + (1-M)V_s \ &\Rightarrow\ rW = MrV_b + (1-M)rV_s \\ &\Rightarrow\ rW = M(1-M)[u(q) + V_s - V_b] + (1-M)M[-c(q) + V_b - V_s] \\ &\Rightarrow\ rW = M(1-M)[u(q) - c(q)] \end{aligned}$$

So the welfare maximizing quality is

$$q^* = \underset{q}{argmax}\ rw \ \Rightarrow\ u'(q^*) = c'(q^*)$$

Special Note: Divisible Money

Now, consider a modification to the model where money is divisible. Let $F(m)$ describe the exogenous distribution of money holdings and let $V(m)$ be the value of holding money. Denote m_b and m_s as the money holdings of buyers and sellers. Recall that d is threat points. The modified Nash Bargaining problem is then

$$(q, d) = \underset{q',d'}{argmax}\ [u(q') + V(m_b - d')]\,[-c(q') + V(m_s + d')]$$

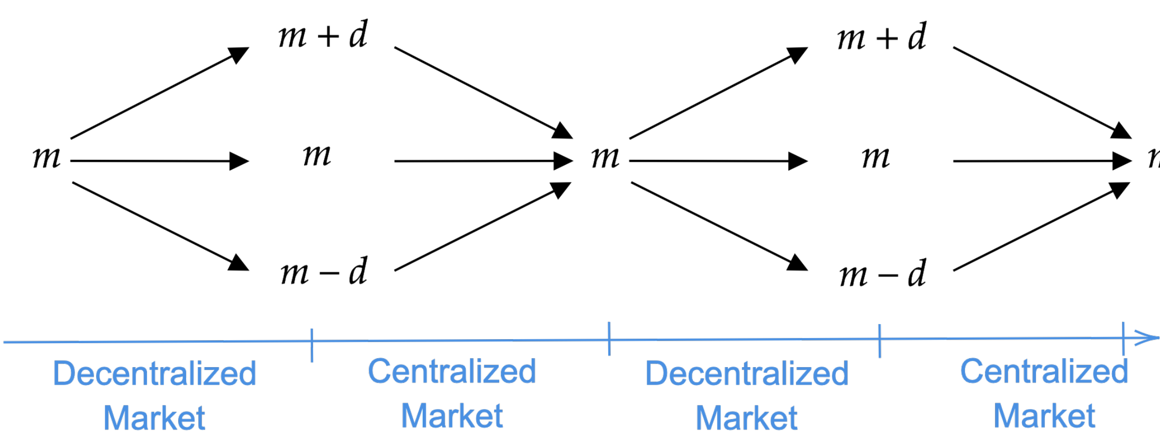
The strategic bargaining flow equation is

$$\begin{aligned} rV(m) = &\int \big\{u(q(m, m_s)) + V(m - d(m, m_s)) - V(m)\big\} dF(m_s) \\ &+ \int \big\{ - c(q(m_b, m)) + V(m + d(m_b, m)) - V(m)\big\} dF(m_b) \end{aligned}$$

With some work, we can once again show that $q^E < q^*$

10.4 Lagos & Wright (2005)

Lagos and Wright proposed a slightly different framework. Consider the same infinite horizon model, but now each period is split into two periods: Decentralized Market and Centralized Market.



As illustrated in the graph above, in the decentralized market, buyers and sellers bargain. Specifically, buyers propose take-it-or-leave-it offers to the seller. After the bargain ends, the new "buyers" and "sellers" enter the centralized market, and the next period begins. In this case, the Nash Bargaining problem is:

$$\max \ [u(q)+V_s]^1 [-c(q)+V_b]^0 \ \ s.t. \ -c(q)+V_b = V_s$$

Remark: Reading past notes, one can see that this model was covered in detail in certain years. But this model was only loosely covered in our lectures and the discussion was only about 10 minutes. The take away from this model is to structurally separate the extensive margin and the intensive margin.

10.5 Mortenson-Pissarides Model

The last model we will discuss in this course is the Morteson-Pissarides *Job Creation and Job Destruction* model. Consider the following environment:

- Infinite Horizon $t = 0, 1, 2, \ldots, \infty$
- Measure 1 of workers
- Infinite number of firms
- Bilateral random matching of meetings (endogenous probability(
- Let y be the output produced in the meeting
- Let b be the unemployment benefit
- Let δ be the probability of exogenous breakdown/separation
- Workers are risk-neutral (solves $\sum_{t=0}^{\infty} \beta^t c_t$)
- Firms solves $\max E\left[\sum_{t=0}^{\infty} \beta^t(\pi_t - k)\right]$
    - k is the cost of posting vacancy
    - $\pi_t = y_t - w_t$ where w_t is the wage from the worker-firm Nash Bargaining
- Let u_t be the unemployment rate
- Let v_t be the vacancy rate

- Let $m(u_t, v_t)$ be a matching function that determines the probability of the bilateral meeting
  - $m(u_t, v_t)$ is increasing, concave, and homogeneous of degree 1
  - Assume that $m(0, v_t) = m(u_t, 0) = 0$
  - Unemployment worker meets a vacancy with probability $\frac{m(u_t, v_t)}{u_t} = m\left(1, \frac{v_t}{u_t}\right) = m(1, \theta)$
  - θ is the "tightness" of the market (think of it as available jobs for each unemployed worker)
  - Firms meet workers with probability $\frac{m(u_t, v_t)}{v_t} = m\left(\frac{u_t}{v_t}, 1\right) = m\left(\frac{1}{\theta}, 1\right)$

10.5.1 Nash Bargaining (Intensive Margins)

The elements of this Nash Bargaining problem are:

- Let $W(w)$ be the value of being employed at wage w
- Let U be the value of being unemployed
- Let $J(y - w)$ be the value of firm employing a worker
- Let V be the value of having a vacancy
- Let S be the surplus of the meeting $S = W(w) - U + J(y - w) - V$
- Let α be the bargaining power of workers

So the Nash Bargaining problem is:

$$w = \underset{w'}{argmax}\ \left[W(w') - U\right]^{\alpha} \left[J(y - w') - V\right]^{1-\alpha}$$

The F.O.C. is

$$\alpha \left[W(w') - U\right]^{\alpha-1} \left[J(y - w') - V\right]^{1-\alpha} W'(w') - (1 - \alpha) \left[W(w') - U\right]^{\alpha} \left[J(y - w') - V\right]^{-\alpha} J'(y - w') = 0$$

Simplifying this, we get

$$\alpha \left[J(y - w') - V\right] W'(w') = (1 - \alpha) \left[W(w') - U\right] J'(y - w')\ (\star)$$

10.5.2 Value Functions (Extensive Margins)

Assuming that $S > 0$, the value functions of the workers and firms are

$$W(w) = \frac{1}{1+r}\left[w + \underbrace{\delta U}_{\substack{\text{Probability of Separation} \\ \text{times Value of Unemployed}}} + \underbrace{(1-\delta)W(w)}_{\substack{\text{Probability of NO Separation} \\ \text{times Value of Employed}}}\right]$$

$$J(y-w) = \frac{1}{1+r}\left[y - w + \delta V + (1-\delta)J(y-w)\right]$$

Doing the same math we have always done for value functions, we get

$$rW(w) = w + \delta\left[U - W(w)\right] \;\Rightarrow\; W(w) = \frac{w+\delta U}{r+\delta} \;\Rightarrow\; W'(w) = \frac{1}{r+\delta}$$

$$rJ(y-w) = y - w + \delta\left[V - J(y-w)\right] \;\Rightarrow\; J(y-w) = \frac{y-w+\delta V}{r+\delta} \;\Rightarrow\; J'(y-w) = \frac{1}{r+\delta}$$

Seeing that $W'(w') = \frac{1}{r+\delta} = J'(y-w')$, we can rewrite equation $(\star)$ as

$$\alpha\left[J(y-w') - V\right]\cancel{W'(w')} = (1-\alpha)\left[W(w') - U\right]\cancel{J'(y-w')}$$

$$\Rightarrow\; \alpha\left[\underbrace{J(y-w') - V + W(w') - U}_{=S}\right] = W(w') - U$$

$$\Rightarrow\; \alpha S = W(w') - U,\;\; (1-\alpha)S = J(y-w') - V$$

Don't forget about

$$U = \frac{1}{1+r}\left[b + m(1,\theta)\cdot W + [1 - m(1,\theta)]U\right] \;\Rightarrow\; rU = b + m(1,\theta)(W-U)$$

$$V = \frac{1}{1+r}\left[-k + m\left(\frac{1}{\theta},1\right)\cdot J + \left[1 - m\left(\frac{1}{\theta},1\right)\right]\cdot V\right] \;\Rightarrow\; rV = -k + m\left(\frac{1}{\theta},1\right)(J-V)$$

10.5.3 Equilibrium

Definition: Equilibrium is a list of (W, U, J, V) and w, θ, u, v such that

- w solves the Nash Bargaining Problem given value functions
- Value functions solve the Bellman Equations given w
- Firms have free entry, so $V = 0$

We shall define the flow equation for the surplus S as:

$$rS = r\left[W - U + J - V\right] = \underbrace{y - w - \delta(J-V)}_{r\cdot J} + \underbrace{w - \delta(W-U)}_{r\cdot W} - \underbrace{b + m(1,\theta)(W-U)}_{rU} - \underbrace{0}_{r\cdot V = r\cdot 0}$$

$$= y - b - \delta\underbrace{(W - U + J - V)}_{=S} - m(1,\theta)\underbrace{(W-U)}_{=\alpha S} = y - b - [\delta + \alpha m(1,\theta)]S$$

Move all terms with S to the left side, we can define $F(\theta)$ with

$$[r + \delta + \alpha m(1,\theta)]S = y - b \;\Rightarrow\; S(\theta) = \frac{y-b}{r+\delta+\alpha m(1,\theta)} \equiv F(\theta)$$

From $V = 0$, we get

$$k = m\left(\frac{1}{\theta}, 1\right)(J - V) = m\left(\frac{1}{\theta}, 1\right)(1-\alpha)S \;\Rightarrow\; S(\theta) = \frac{k}{(1-\alpha)m\left(\frac{1}{\theta}, 1\right)} \equiv G(\theta)$$

In equilibrium, we must then have

$$\frac{y-b}{r+\delta+\alpha m(1,\theta)} = F(\theta) = S(\theta) = G(\theta) = \frac{k}{(1-\alpha)m\left(\frac{1}{\theta}, 1\right)}$$

From this condition, we get $\theta^* = \frac{v}{u}$ (optimal tightness) and $S^* = S(\theta^*)$ (optimal surplus).

In steady-state, we must have

$$\underbrace{m(1,\theta)u}_{\substack{\text{Joint mass}\\ \text{of an unemployed}\\ \text{worker getting a job}}} = \underbrace{(1-u)\delta}_{\substack{\text{Joint mass}\\ \text{of an employed}\\ \text{worker leaving}}} \;\Rightarrow\; u = \frac{\delta}{\delta + m(1,\theta)}$$

Since $\theta = \frac{v}{u}$, it must be that $v = \theta \cdot u = \dfrac{\theta\delta}{\delta + m(1,\theta)}$, where v is the mass of vacancies in steady-states. Since $V = 0$, we also know

$$rJ = y - w + \delta(V - J) \;\Rightarrow\; w = y - rJ - \delta(J - V) = y - r(\underbrace{J - V}_{V=0}) - \delta(J-V)$$

$$= y - (r+\delta)(J-V) = y - (r+\delta)(1-\alpha)S(\theta^*)$$

$$w = y - (r+\delta)(1-\alpha)S(\theta^*)$$

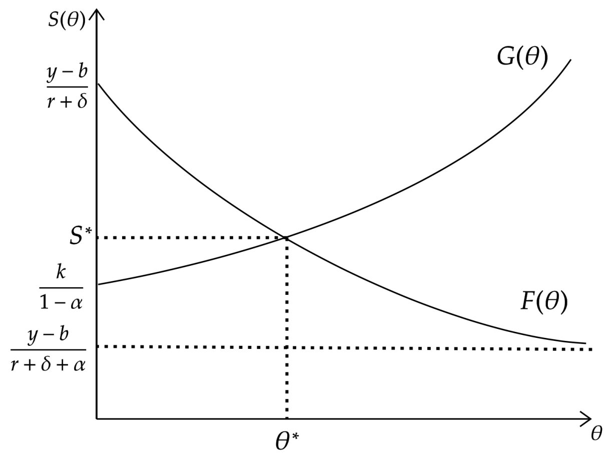
So how do we qualify the steady-state tightness θ^*? Well we know that:

$$F(\theta) = \frac{y-b}{r+\delta+\alpha m(1,\theta)};\ F'(\theta) = -\frac{m_2(1,\theta)}{[r+\delta+\alpha m(1,\theta)]^2} < 0$$
$$F(0) = \frac{y-b}{r+\delta};\ F(\infty) = \frac{y-b}{r+\delta+\alpha}$$
$$G(\theta) = \frac{k}{(1-\alpha)m\left(\frac{1}{\theta},1\right)};\ G'(\theta) = \frac{m_1\left(\frac{1}{\theta},1\right)}{[(1-\alpha)m\left(\frac{1}{\theta},1\right)]^2} > 0$$
$$G(0) = \frac{k}{1-\alpha};\ G(\infty) = \infty$$

Since one is a strictly decreasing function and the other is a strictly increasing function, we get $\theta^* > 0$ if and only if

$$\frac{k}{1-\alpha} < \frac{y-b}{r+\delta}$$
$$\Rightarrow k < \frac{(1-\alpha)(y-b)}{r+\delta}$$

The graph to the right depicts θ^* and S^*, which gives us the bargaining equilibrium.



10.5.4 Comparative Statics

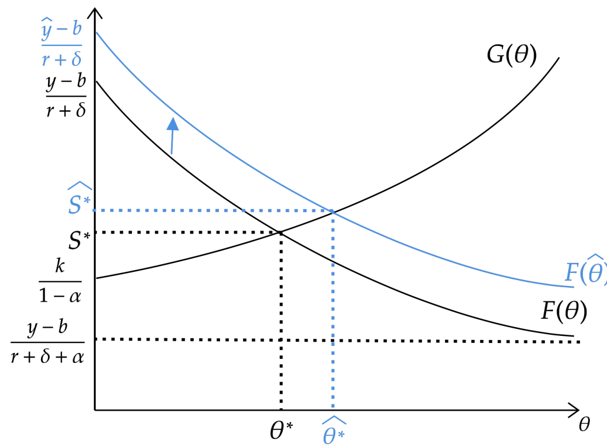
Now that we have qualified our equilibrium, we show study it using comparative statics.

Case 1: Increase in y

Recall that

$$F(\theta) = \frac{y-b}{r+\delta+\alpha m(1,\theta)}$$
$$G(\theta) = \frac{k}{(1-\alpha)m\left(\frac{1}{\theta},1\right)}$$

so an increase in y means $F(\theta)$ increases and $G(\theta)$ remains the same. This leads to an increase in θ^* and S^*.



As for the labor market (extensive margins), we have

$$u = \frac{\delta}{\delta + m(1,\theta)} \;\Rightarrow\; \frac{du}{d\theta} = \frac{-\delta m_2(1,\theta)}{[\delta + m(1,\theta)]^2} < 0$$

$$v = \frac{\delta\theta}{\delta + m(1,\theta)} \;\Rightarrow\; \frac{dv}{d\theta} = \frac{\delta[\delta + m(1,\theta)] - \delta\theta m_2(1,\theta)}{[\delta + m(1,\theta)]^2} = \frac{\delta[\delta + m(1,\theta) - \theta m_2(1,\theta)]}{[\delta + m(1,\theta)]^2}$$

Notice that since $m(u,v)$ is homogeneous of degree 1, $m(1,\theta) = 1 \cdot m_1(1,\theta) + \theta m_2(1,\theta)$, so we can rewrite the second equation and get

$$\frac{du}{d\theta} = \frac{-\delta m_2(1,\theta)}{[\delta + m(1,\theta)]^2} < 0$$

$$\frac{dv}{d\theta} = \frac{\delta[\delta + m_1(1,\theta)]}{[\delta + m(1,\theta)]^2} > 0$$

So when y increases, v increases and u decreases, which is consistent with the predictions of the Beveridge curve. Equivalently, when productivity increases, firms are willing to hire more people. As such, worker bargaining power increases and while firm bargaining power decreases, which is the same as $(1-u)$ increasing and $(1-v)$ decreasing, and w will increase.

Case 2: Increase in unemployment benefit b

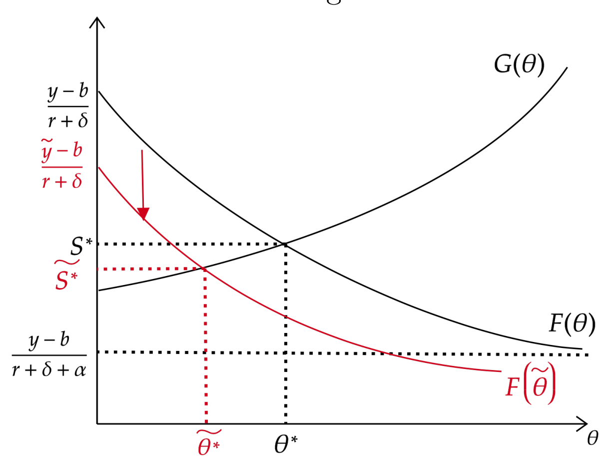
Similarly, since

$$F(\theta) = \frac{y - b}{r + \delta + \alpha m(1,\theta)}$$

$$G(\theta) = \frac{k}{(1-\alpha)m\left(\frac{1}{\theta}, 1\right)}$$

so an increase in y means $F(\theta)$ decreases and $G(\theta)$ remains the same. This leads to an decrease in θ^* and S^*.

One can see that increase in b increases reservation wage, so u increases, holding labor demand constant we must then have v increases (but less than increase in u), and the effect on w is thus ambiguous.



This is the end of this lecture note. Good luck on finals and prelims!