[SP26] ECN 812B Recitation 7

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1 Concepts this Week

- Repeated games: A stage game (players, strategy sets, **AND** payoffs) that is repeated finitely/infinitely many times
- When there are multiple NEs in the stage game, a non-NE strategy profile is potentially sustainable. i.e., playing non-NE strategies can be a best response when you calculate the total payoff.
  - Recall that strategies are specified combinations of actions at each *information set (not node)*, so to make a strategy in a repeated game setting well-defined, we need to specify the history of the game as part of the strategy.
- Nash Reversion Strategy: A strategy that rewards cooperation with future cooperation, and punishes non-cooperation with NE strategies that are bad for the opponent for the remainder of the game.
- If not using NRS, we need to check the *one-shot deviation principle* to make sure that the punishment makes sense. For example, can playing a non-NE as punishment be a sustainable equilibrium strategy?
- **Principle of Optimality (O&R Lemma 153.1):** A strategy profile is a SPNE of the δ-discounted infinitely repeated game of stage game Γ *if and only if* no player can gain by deviating in a single period after any history. Always check deviation from collaboration, and then make sure to check deviation from punishment.
- **Folk Theorem**: As $\delta \to 1$, any payoff v that is in the convex hull V of the stage game payoffs is feasible as an average payoff in a repeated game PS-SPNE if

$$v_i > \min_{s_{-i}}\{\max_{s_i} u_i(s_i, s_{-i})\}$$

  If $v_i = \min_{s_{-i}}\{\max_{s_i} u_i(s_i, s_{-i})\}$, we need to make sure that the player enforcing the punishment does not have an incentive to deviate and not punish. i.e., the punishment cannot be too harsh on the person enforcing it.

- The payoff v described in Folk theorem is the “average payoff” calculated as (let v^t denote payoff in period t):

$$\begin{aligned} v_i + \delta \cdot v_i + \delta^2 \cdot v_i + \cdots &= v_i^1 + \delta \cdot v_i^2 + \delta^2 \cdot v_i^3 + \cdots \\ \Longleftrightarrow \frac{v_i}{1-\delta} &= v_i^1 + \delta \cdot v_i^2 + \delta^2 \cdot v_i^3 + \cdots \\ \Longleftrightarrow v_i &= (1-\delta)(v_i^1 + \delta \cdot v_i^2 + \delta^2 \cdot v_i^3 + \cdots) \end{aligned}$$

  So whatever you solve as the total expected payoff from the SPNE, the average payoff (used in Folk theorem) is the total payoff multiplied by $(1-\delta)$.

2 Learning by Doing

1. (Modified from MSU Prelim SS 2016 Part II Q1) Let $x \geq 0$, consider the following prisoner's dilemma game repeated $T \in \mathbb{N}$ times with discount rate $\delta \in (0, 1)$.

		P_2: C	P_2: D
P_1	C	$3, 3$	$1, x$
	D	$5, 1$	$0, 0$

   (a) What is the range of x for which we can find a δ that sustains a non-NE profile being played in every period until the T^{th} period?

   (b) Suppose that the stage game is now repeated infinitely many times. Let x be in the range found in (a). What is the Nash Reversion Strategy that can sustain the profile (C, C) being played in every stage game.

(c) Without doing any formal analysis, what is the relationship between δ and x if we want to sustain the profile (C, C) being played in every stage game?

(d) Find the smallest δ such that there is a subgame perfect equilibrium of the infinitely repeated game with discounting in which both players play C every period along the equilibrium path. Is your answer in (c) consistent with this result?

2. (Modified from MSU Prelim May 2023) Consider a Cournot model with three firms with 0 marginal cost. The inverse demand function in the market is $P(Q) = 12 - Q$. The Cournot outcome is $(q_1, q_2, q_3) = (3, 3, 3)$ with profits $(\pi_1, \pi_2, \pi_3) = (9, 9, 9)$. Let this game be played with infinite repetition, we want to consider the possibility of collusion that sustains the monopoly outcome in the market with repeated interactions. Specifically, construct an SPNE such that each firm produces $\frac{1}{3}$ of the monopoly quantity and use the stage game Nash equilibrium as the punishment strategies (Nash reversion). Find the condition on δ, such that the SPNE can be sustained.

3. (MSU Prelim FS 2017 Q3) Consider the following buyer-seller game: the buyer (B) decides whether to buy (b) or not (n) whereas the seller (S) decides whether to provide high quality (H) or low quality (L). The decisions are simultaneous. The payoffs are as follows:

		B: b	B: n
S	H	$2, 3$	$0, 0$
	L	$3, 2$	$0, 0$

(a) What is the Nash Equilibrium of this game and what are the minmax payoffs of the two players?

Next, consider an infinitely repeated game where the above stage game is played in each period. Both parties have a common discount factor $\delta \in (0, 1)$. Answer the following questions:

(b) Sketch the set of feasible payoffs and the set of average payoffs that can be sustained in an SPNE as $\delta \to 1$ (as per the folk-theorem discussed in class).

As you may see from part (b) above, folk theorem would imply that there exists an SPNE for δ sufficiently large where in each period the buyer buys and the seller provides high quality. That is, we can have $(2, 3)$ be the of every stage game in SPNE. Consider a set of strategies where (H, b) is played in each period but if the seller "cheats" the buyer by choosing L, the buyer expects the seller to continue to choose L until the buyer punishes him by not buying. Formally, the strategies are as such:

Buyer: In period 1, play b. In any period $t > 1$, play n if the last period's $(t - 1)$ play is (L, b). Otherwise, always play b.

Seller: In period 1 play H. In any period $t > 1$, play L if the last period's $(t - 1)$ play is (L, b). Otherwise, always play H.

(c) Find the value of minimum δ such that the above strategies constitute an SPNE. [*Hint:* Note that δ has to be high enough so that seller has incentive to choose H and also the buyer has incentive to choose n if seller has deviated in the past.]

4. (Columbia Prelim 2011) Recall that the stage game of Matching Pennies is characterized by the following payoff matrix:

P_1 \ P_2	Heads	Tails
Heads	$-1, 1$	$1, -1$
Tails	$1, -1$	$-1, 1$

(a) What does the folk theorem imply for the infinitely repeated Matching Pennies game? Explain your answer fully.

(b) How would your answer change if the stage-game is changed as follows:

P_1 \ P_2	Heads	Tails	Other
Heads	$-1, 1$	$1, -1$	$0, 0$
Tails	$1, -1$	$-1, 1$	$0, 0$
Other	$0, 0$	$0, 0$	x, x

3 Go the Extra Mile

1. (2019 Practice Exam Q3) The simultaneous-move game below is played twice, with the outcome of the first stage observed before the second stage begins. There is no discounting. Can the payoff $(4,4)$ be achieved in the first stage in a pure-strategy subgame-perfect Nash equilibrium? If so, give strategies that do so. If not, prove why not.

Player 2

Player 1		L	C	R
	T	$3,1$	$0,0$	$5,0$
	M	$2,1$	$1,2$	$3,1$
	B	$1,2$	$0,1$	$4,4$

2. (MSU Prelim FS 2014, Part II Q2) Consider the following stage games:

(*Game* 1)

P_2

P_1	C	D
C	$4,4$	$0,5$
D	$6,0$	$1,1$

(*Game* 2)

P_2

P_1	C	D
C	$4,4$	$0,6$
D	$5,0$	$1,1$

(a) Consider Game 1. Find the Nash equilibrium and minmax payoff. In a labeled diagram, sketch the set of individually rational payoffs.

(b) Suppose two players play only Game 1 (or only Game 2) infinitely many times, with common discount factor and perfect monitoring. What is the minimum δ that will allow the existence of a subgame perfect Nash equilibrium in which the players play $(C;C)$ along the equilibrium path (consider grim-trigger strategies)?

(c) Now suppose the two players will play both games simultaneously in each period. Find the minimum δ that will allow the existence of a subgame perfect Nash equilibrium in which they play $(C;C)$ in both games along the equilibrium path.

(d) When is it easier to sustain cooperation—when they play Game 1 only or when they play both games simultaneously? Provide a brief intuition for your result.

3. (MSU 2021 Exam 2 Q2) Consider a game with two players ($i = 1, 2$) with the following action and payoff space:

		P_2		
		L	C	R
P_1	U	$3, 3$	$0, 4$	$-\frac{3}{2}, -1$
	M	$4, 0$	$1, 1$	$-1, -1$
	D	$-1, -\frac{3}{2}$	$-1, -1$	$-2, -2$

(a) Suppose that the two players move sequentially. Player 1 moves first with a publicly observable action, followed by Player 2. Find all pure strategy SPNE of this sequential game.

(b) Suppose now that the two players move simultaneously. They play the simultaneous move game twice and both possess a common discount factor $\delta \in (0, 1)$. Find all pure strategy SPNE of this repeated game.

(c) Use the Folk Theorem to characterize and/or draw the set of feasible average discounted payoffs in an SPNE, clearly identifying the minmax strategies and payoffs.

(d) Suppose now that the game is repeated infinitely. Define a grim trigger strategy for each player such that (U, L) is the outcome of the infinitely repeated game in each period, with a threat of minmax punishment. What is the lowest δ such that (U, L) is realized every period?

(e) Is the strategy you specified in the previous part an SPNE? Why or why not? If not, specify another grim trigger strategy with (U, L) as an SPNE outcome of the infinitely repeated game. What is the lowest δ such that this strategy can be supported as an equilibrium?

4. (MSU 2016 Final Q3) Consider the following stage game with three players all having action sets $\{A, B\}$. Suppose P_3 chooses the matrix:

A:

		P_2	
		A	B
P_1	A	1,1,1	0,0,0
	B	0,0,0	0,0,0

B:

		P_2	
		A	B
P_1	A	0,0,0	0,0,0
	B	0,0,0	1,1,1

Suppose the above stage game is repeated infinitely many times. The discount factor is $\delta \in (0, 1)$ for all players. Can there exist a SPNE of this game where a player's average payoff is in $[0, \frac{1}{4})$? If so, state the strategy profiles that sustain such a payoff in SPNE. If not, give a proof. [*Hint*: No matter what strategy profile is chosen, note that in any period at least two players would play the same action with at least $1/2$ probability. Using this fact, try to deduce what could be the minimum payoff a player can get in a SPNE.]