[SP26] ECN 812B Recitation 9

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1 Concepts this Week

- Continuous Type BNE: It's the same thing as discrete type, always remember to specify your strategies to be contingent on each agent's own known types.
  - In auctions, the strategies are bids as a function of own/known valuations. When you check for deviations, differentiate on the bid b_i and not the strategy $b(v_i)$. One common mistake is to take the derivative of $b(v_i)$ with respect to v_i.
  - Bidding strategies often implicitly assume monotonicity, i.e., $\frac{d}{dv_i}b(v_i) > 0$. Sometimes this is not given and you may need to check for that.
  - The BNE maximizes one's own expected utility given own valuation v_i.
- Adverse Selection (Preview): The phenomenon where asymmetric information leads to inefficiency, and sometimes the complete "unravelling" of the market.
  - Instead of making decisions using the expected value of a good, buyers/sellers make decisions using the *conditional expected value* of a good.
  - For example, if the buyers only know a good is valued at $v \sim U[0,1]$ but seller knows thee exact value and will only sell if $p \geq v$. Then the expected value of the good priced at v is $E[v \mid v \leq p] = \frac{p}{2}$ instead of $E[v] = \frac{1}{2}$.

2 Learning by Doing

1. (Modified MSU May 2022) A seller wants to sell a single object and has two interested buyers. The buyers' valuations of the object are independently and uniformly distributed on $[0, 1]$, and each buyer privately knows their own valuation. The seller's value of the object is 0. Suppose the seller adopts the following selling strategy. She approaches one of the buyers (chosen at random) and makes a take-it-or-leave-it offer at a fixed price p_1. If the first buyer accepts the offer, the object is sold to them at the offered price. If the first buyer declines the offer, the seller then approaches the other buyer with a take-it-or-leave-it offer at a fixed price p_2. If the second buyer accepts the offer, the object is sold to them at the offered price. If neither buyer accepts, then the seller keeps the object.

   (a) Derive the optimal values of p_1 and p_2. What is the seller's expected revenue from this selling strategy?

(b) Suppose that the seller can also run a first-price auction (without reservation price) instead. Standard theory tells us that in BNE, the buyer's equilibrium strategy is $b_i(v_i) = \frac{v_i}{2}$ and the seller's expected revenue is $\frac{1}{3}$ (This outcome was the answer to the original part (b) of the question). Between this and the outcome in part (a), which one is more efficient? Does the seller always prefer the more efficient option?

2. (MSU Final 2024) Consider a first-price sealed-bid auction of a single indivisible good with two bidders. Each bidder values the good at either v_L or v_H. The bidders know their own valuation of the good but not the other bidder's valuation. The common prior is that $P(v_i = v_L) = \lambda$. In case of a tie, a fair two-sided coin is flipped to decide the winner of the auction.

    (a) Show that there is no symmetric pure strategy Bayes-Nash equilibrium in this game.

Suppose that, in equilibrium, a bidder of value v_L still bids $b_L = v_L$. Answer parts (b) through (d) to find the symmetric optimal bidding strategy of bidders with v_H.

(b) What is the maximization problem of a bidder of value v_H?

(c) Show that bidder i of value v_H is a mixed strategy following the distribution with cdf $F(b_i)$, where $F(b_i)$ satisfies the equality

$$-\int_{v_L}^{b} \frac{d}{db_i}\left[(v_H - b_i)(1 - F(b_i))\right] db_i = \int_{v_L}^{b} \frac{1}{1-\lambda}\, db_i.$$

(d) Solve for $F(b)$ and argue that the support of b is $[v_L, \mu]$ where $\mu \in (v_L, v_H)$. What happens to μ as λ (the probability that the other bidder is type v_L) decreases?

3. (MSU Exam 2 2018) Consider a **second-price auction** between two players, where bidder i values the object at v_i, but unlike the standard private value model, assume that the valuations v_1 and v_2 are **correlated**. In particular, assume that each bidder **privately** receives a signal $s_i \in \mathcal{R}$, and value of the good to bidder i is:

$$v_i = \alpha s_i + \beta s_{-i}$$

where $\alpha \geq \beta \geq 0$. (Notice that the standard private value model assumes $\alpha = 1$ and $\beta = 0$). Suppose that both s_1 and s_2 are distributed independently and uniformly over $[0, 1]$. Find a symmetric BNE where each bidder's bid is proportional to her own signal (i.e., in BNE, bidder i bids according to $b_i(s_i) = cs_i$ for some constant $c > 0$. You need to find c).

4. (MSU 2023 FS Q4) monopolist faces a customer whose valuation for the product is his private information. The monopolist's production cost is 0, and the customer's valuation v is known to be distributed as $U[0, 1]$.

   (a) What is the profit maximizing price for the monopolist? What is the expected profit? Show optimal price and associated profit in the graph drawn above. How much consumer surplus does a customer with valuation v earns under monopoly pricing?

(b) Now suppose the customer has an option to fully reveal her valuation to the monopolist. The game proceeds in two steps: first, the customer decides whether or not to reveal v; next, monopolist offers a price based on the customer's disclosure decision. Note that in any equilibrium of this game, the monopolist charges a price $p = v$ to a customer who discloses her valuation, and an uniform price p_0 (say) to all customer who does not disclose her type. What is the minimum value of p_0 that can be sustained in an equilibrium of this game? Can the customer be better off by having such an disclosure option?

5. (MSU 2020 Exam 3 Q1) Each person may have a privately known pre-existing health condition with the severity of the condition uniformly distributed between 0 and 1 in the population. Any person without a health insurance gets a payoff 0. A person with condition severity (type) θ will derive a payoff θ from buying a health insurance. The health insurance can only charge a uniform price p to all individuals, but the cost of insuring each individual is different: The cost of insuring a type θ individual is $\theta/2$.

   (a) What is the profit function for a monopoly health insurance provider?

   (b) What is the profit-maximizing price when there is a single health insurance provider?

   (c) Suppose two health insurance providers simultaneously announce their prices, and individuals choose to buy from one insurer. What is the equilibrium price?

   (d) Suppose a law penalizes any individual who does not buy a health insurance by incurring a cost of $1/4$. (The penalty becomes government revenue). What is the profit-maximizing price when there is a single health insurance provider?

3 Go the Extra Mile

1. (MSU 2017 Exam 1 Q3) Suppose that there are n bidders bidding for an object in a sealed-bid all-pay auction. That is, players submit their bids simultaneously; only the highest bidder is awarded the object, but each player must pay his own bid irrespective of whether he wins or not. Bidder's values are drawn independently from the uniform distribution on $[0; 1]$ and are privately known.

   (a) Find a symmetric Bayesian Nash equilibrium of this game in monotonically increasing bidding strategies. Calculate the seller's expected profit.

   (b) Now suppose the object has a common value: all bidders value it at 1. The auction rule is still the same as above. Find a symmetric Bayesian Nash equilibrium of this game. Calculate the seller's expected profit. [Hint: Note that this is a game of *complete information.* Also, first argue that a pure strategy NE cannot exist and then try to compute a mixed strategy NE where a strategy is simply a probability distribution over $[0; 1]$.]

2. (MSU 2021 Exam 1 Q2) Consider two firms competing in Stakelberg fashion. Firm 1 is the leader who chooses output q_1. Firm 2 observes q_1 before choosing its output q_2. Fixed costs are 0 to both firms. Firm 1 has constant marginal cost $c_1 > 0$, and this is known to both firms. Firm 2 might be a low-cost firm, given by constant marginal cost $c_L > 0$, or a high-cost firm, given by constant marginal cost $c_H > 0$. Firm 2 knows its type, but firm 1 only knows that firm 2 is a low-cost firm with probability $\frac{1}{2}$. Suppose inverse market demand is given by $p = 100 - Q$ where Q is the total output ($Q = q_1 + q_2$) in the market. Suppose $c_1 = 6, c_L = 4$, and $c_H = 12$.

   (a) Formulate this situation as a Bayesian game.

   (b) Find a Bayesian Nash Equilibrium. [Note: Remember the distinction between equilibrium strategies and outcomes.]

   (c) Compute the equilibrium profit for each firm.

3. (MSU 2016 Exam 2 Q3) Consider the following model of trade under adverse selection. A seller S wants to sell a product to a buyer B. The product's quality $q \in [0, 1]$ is

known to S but not to B. Assume that the buyer's prior belief about q is uniform on $[0, 1]$. When the product quality is q, its value to S is $q/3$ whereas its value to B is exactly q.

(a) Show that no quality gets sold (expect for 0) in the competitive equilibrium with price zero (where price is equal to the expected quality available on the market).

(b) Now consider the following trading rule: For every qualily $q \in Q$, S posts a take-it-or-leave-it price p and may choose to report the true quality. That is, given quality $q \in Q$, S sends a message $m \in \{q, \emptyset\}$ where $m = \emptyset$ means S chooses not to reveal the quality. Upon observing $(\bar{p}, m)$, B decides whether to buy or not at price $\bar{p}$. The seller's payoff is $p - q/3$ and the buyer's payoff is $q - p$. Assume that the buyer buys the product as long as p does not exceed the product's expected quality. Show that there exists a PBE where S always reveals the true quality and all qualities are traded at $p = q$. Clearly state the strategies of S and B and the associated belief of the buyer. [*Hint:* The off-equilibrium belief must be such that upon observing $m = \emptyset$, B forms a very unfavorable opinion about the underlying quality.]

(c) Consider a further variation of the model described so far. Suppose that S is informed about the quality with a probability $\alpha < \frac{1}{2}$ and he cannot report q to B. What is the competitive equilibrium price in this situation and what quality levels, if any, are traded at equilibrium? (You should be able to derive the subset of qualities that are traded in equilibrium.)