[SP26] ECN 812B Recitation 10

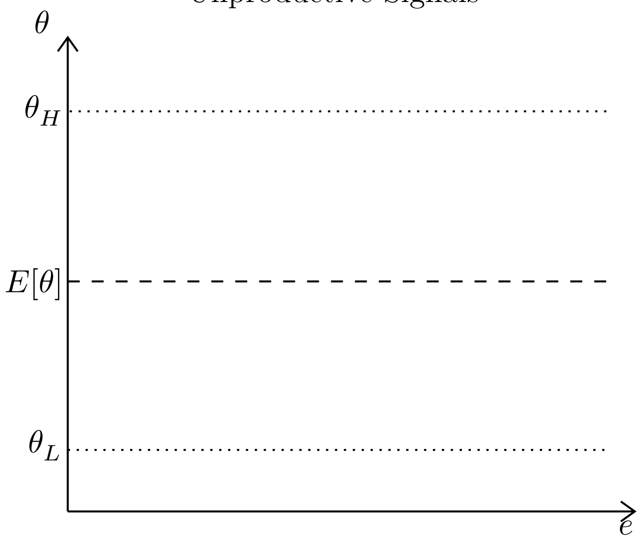
Willy Chen

April 3, 2025

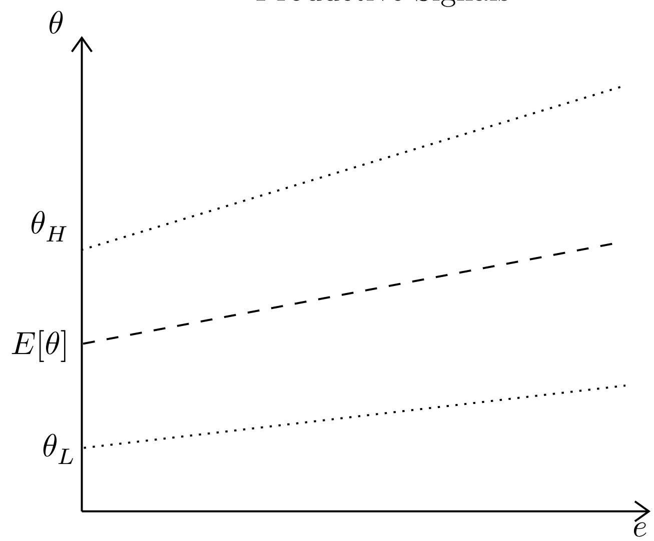
1 Concepts this Week

- Adverse Selection: The phenomenon where asymmetric information leads to inefficiency, and sometimes the complete "unravelling" of the market.
  - Instead of making decisions using the expected value of a good, buyers/sellers make decisions using the *conditional expected value* of a good.
  - For example, if the buyers only know a good is valued at $v \sim U[0,1]$ but seller knows thee exact value and will only sell if $p \geq v$. Then the expected value of the good priced at v is $E[v \mid v \leq p] = \frac{p}{2}$ instead of $E[v] = \frac{1}{2}$.
  - Sometimes market power is added to the questions, so one needs to be aware of the competitive environment.
  - Adverse selection need not be about buyers/sellers of goods. The model also applies to the labor market.
- Signaling: An application of wPBEs and PBEs where agents with private information can give a (costly) signal to reveal information.
  - Generally separated into (1) Separating Equilibrium and (2) Pooling Equilibrium
  - Incentive Compatibility: Each agent has the incentive to do what you want them to do. In a separating equilibrium, this means none of the types have an incentive to pretend to be another type.
  - Individual Rationality: Each agent has no incentive to **not** participate.

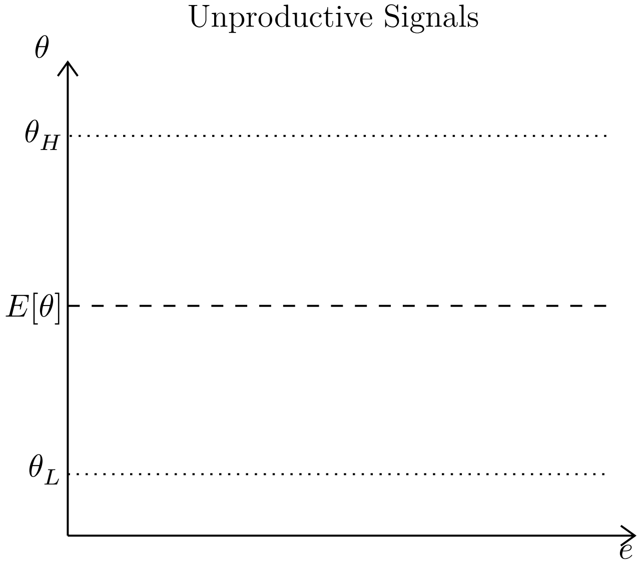
Unproductive Signals



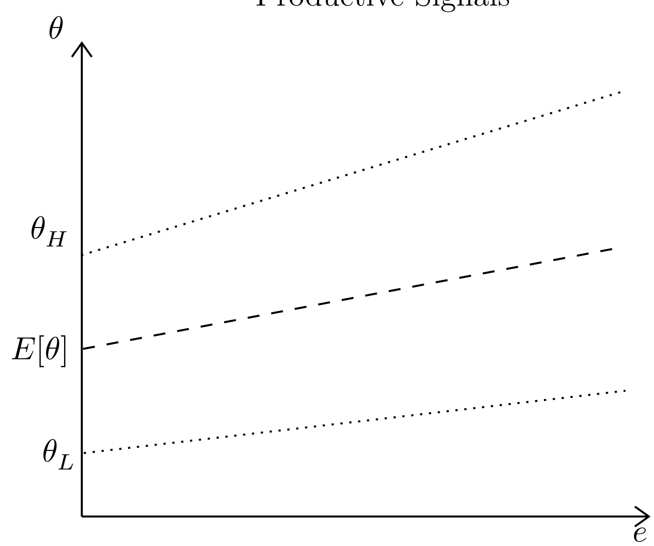
Productive Signals



Unproductive Signals



Productive Signals



2 Learning by Doing

1. (MSU Prelim SS 2023 Q4, SS 2016 P2 Q3) Consider the Akerlof (1970) used car model. Each seller knows the quality of the car denoted by θ. Prospective buyers only know the distribution of θ, which is uniformly distributed on $[0, 1]$. If a prospective buyer purchases a car of quality θ at the price of p, his utility is given by $\theta - p$. If the buyer does not make a purchase, his utility is zero. Buyers are risk-neutral. There is a large number of prospective used car buyers (many more than potential sellers), which simply means that the market price is determined by the expected value of cars in the market to consumers. In contrast, a seller who has a used car of quality θ has the reservation utility of $r(\theta) = k\theta$ where $k \in [0, 1)$; he will only sell if the price is greater than or equal to $k\theta$.

   (a) Suppose that $k = \frac{1}{3}$. Is there an efficient market equilibrium? If so, describe it. If there is not, describe all inefficient equilibria.

   Solution.

   The seller will sell if $p \geq \frac{\theta}{3}$, so the buyer's expected utility from buying, in equilibrium, is:

$$E\left[\theta - p \middle| p \geq \frac{\theta}{3}\right] = E\left[\theta \middle| 3p \geq \theta\right] - p$$

   If $3p \geq 1$, then $E\left[\theta - p \mid p \geq \frac{\theta}{3}\right] = \frac{1}{2} - p \geq 0 \;\Rightarrow\;$ In EQ, $p = \frac{1}{2}$.
   If $3p < 1$, then $E\left[\theta - p \mid p \geq \frac{\theta}{3}\right] = \frac{3p}{2} - p = \frac{1}{2}p > 0$.
   So the market is efficient (in that all cars are bought) if $p \in \{\frac{1}{2}\} \cup (0, \frac{1}{3})$.
   As such, in equilibrium, all cars are sold, with $p \in \{\frac{1}{2}\} \cup (0, \frac{1}{3})$ and the market is efficient.

(b) Suppose that $k = \frac{2}{3}$. Is there an efficient market equilibrium? If so, describe it. If there is not, describe all inefficient equilibria.

Solution.

The seller will sell if $p \geq \frac{2\theta}{3}$, so the buyer's expected utility from buying, in equilibrium, is:

$$E\left[\theta - p \middle| p \geq \frac{2\theta}{3}\right] = E\left[\theta \middle| \frac{3}{2}p \geq \theta\right] - p$$

If $\frac{3}{2}p \geq 1$, then $E\left[\theta - p \mid p \geq \frac{2}{3}\theta\right] = \frac{1}{2} - p < 0$.
If $\frac{3}{2}p < 1$, then $E\left[\theta - p \mid p \geq \frac{2}{3}\theta\right] = \frac{3p}{4} - p = -\frac{1}{4}p < 0$.
As such, in equilibrium no buyers will buy, and so the market completely unravels and the equilibrium is inefficient.

(c) For what values of k does there exist an efficient market equilibrium with the Pareto optimal number of trades?

Solution.

From parts (a) and (b), we know that to get efficient market equilibrium, we must have $E[\theta - p \mid p \geq k\theta] \geq 0$.
If $\frac{p}{k} \geq 1$, $E[\theta - p \mid p \geq k\theta] = \frac{1}{2} - p \geq 0 \iff p \leq \frac{1}{2}$, so we need $k \in (0, \frac{1}{2}]$.
If $\frac{p}{k} \leq 1$, $E[\theta - p \mid p \geq k\theta] = \frac{p}{2k} - p = \left(\frac{1}{2k} - 1\right) p \geq 0 \iff \frac{1}{2k} - 1 \geq 0 \iff k \leq \frac{1}{2}$.
If $k = 0$, $E[\theta - p \mid p \geq 0] = \frac{1}{2} - p$, then buyers buy at $p \leq \frac{1}{2}$.
As such, to get efficient market equilibrium, we must have $k \in \left[0, \frac{1}{2}\right]$.

(d) Assume that $k = \frac{2}{3}$. There is a test service that reveals the quality of a car perfectly. This service is *competitively* supplied at the marginal cost of $\frac{1}{4}$. Describe the market equilibrium.

Solution.

From part (b), we know that at $k = \frac{2}{3}$, buyers do not buy untested cars. By backward induction, if buyers bear the cost of testing $\left(\frac{1}{4}\right)$, they will test if (Notice that in the decision step, the cost of testing is sunk)

$$E\left[\theta - p - \frac{1}{4} \middle| \underbrace{p \leq \theta \leq \frac{3}{2}p}_{\text{Decision Step}}\right] \geq 0$$

If $\frac{3}{2}p \geq 1$, then the SPNE involves testing if and only if:

$$E[\theta \mid p \leq \theta] - p - \frac{1}{4} \geq 0 \iff p \leq \frac{1}{2}$$

But we know that $p \geq \frac{2}{3}$ in this case, so this can't be an equilibrium. Similarly, if $\frac{3}{2}p < 1$, then the SPNE involves testing if and only if:

$$E\left[\theta \middle| \frac{3}{2}p \geq \theta \geq p\right] - p - \frac{1}{4} \geq 0 \iff \frac{p}{4} - \frac{1}{4} \geq 0 \iff p \geq 1$$

But we know that $\frac{3}{2}p < 1$ in this case, so this also can't be an equilibrium. So, by backward induction, the BNE will not involve the buyer paying for the test. As such, sellers must bear the cost of the test $\left(\frac{1}{4}\right)$. Since the information asymmetry is solved via testing, the buyer will buy at $p = \theta$. The sellers then only try to sell cars with θ such that

$$\theta - \frac{1}{4} \geq \frac{2}{3}\theta \iff \frac{1}{3}\theta \geq \frac{1}{4} \iff \theta \geq \frac{3}{4}$$

So in equilibrium, sellers will bear the cost of testing and sell only cars with $\theta \in \left[\frac{3}{4}, 1\right]$ at $p = \theta$. The equilibrium is more efficient than the case without tests, but only $\frac{1}{4}$ of the cars in the market are sold, so it is still inefficient compares to when $k = \frac{1}{3}$.

(e) Continue to assume that $k = \frac{2}{3}$. Now suppose that there is a *monopolistic* supplier of the test (with the same marginal cost of $\frac{1}{4}$ as above). What price the monopolistic test provider would charge for such a service?

Solution.

If sellers bear the cost r of testing, they will sell at $p = \theta$ if and only if

$$p - r = \theta - r \geq \frac{2}{3}\theta \iff \frac{1}{3}\theta \geq r \iff \theta \geq 3r$$

So the demand for tests is $q(r) = P(\theta \geq 3\theta) = 1 - 3r$. The monopolist then solves:

$$\max_r \; q(r) \cdot \left(r - \frac{1}{4}\right) \equiv \max_r \; (1 - 3r)\left(r - \frac{1}{4}\right)$$

The F.O.C. is $1 - 3r - 3r + \frac{3}{4} = 0 \iff 6r = \frac{7}{4} \iff r^* = \frac{7}{24}$, so the monopolist will charge the profit maximizing price $r^* = \frac{7}{24}$.

In equilibrium, sellers sell cars with $\theta \in \left[\frac{7}{8}, 1\right]$ at $p = \theta$. The market equilibrium is thus less efficient than the case in (d).

2. (MSU Prelim FS 2017 Q4) Consider the following variation of the Akerlof's lemon model. A seller sells an object of quality v to a buyer. The seller values an object of quality v at v but the buyer values it at θv where $\theta \in (1, 2)$. Moreover, the seller knows v whereas the buyer only knows that $v \sim U\left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]$ where ε is a parameter between $\left[0, \frac{1}{2}\right]$. The distribution of v is common knowledge. Notice that efficiency requires that all quality types are traded. Suppose that the seller has all bargaining power and sets a price that extracts all surplus from the buyers.

   (a) Find the equilibrium price.

   Solution.

   For any price p, the buyers would only buy if $v \leq p$. So the price must satisfy:

$$\theta E[v \mid v \leq p] = \theta \cdot \frac{\frac{1}{2} - \varepsilon + p}{2} = p \;\Rightarrow\; p^* = \frac{\theta}{2-\theta}\left(\frac{1}{2} - \varepsilon\right)$$

   (b) How does the extent of inefficiency; i.e., the set of quality types that are not traded, vary with ε? Give a brief intuition for your finding.

   Solution.

   To get an efficient equilibrium, all types of cars need to be sold, meaning $p \geq v$, $\forall v$. Specifically, we must have

$$\frac{1}{2} + \varepsilon \leq p = \frac{\theta}{2-\theta}\left(\frac{1}{2} - \varepsilon\right)$$

   meaning that if there is an inefficient equilibrium, $\exists \varepsilon \in \left[0, \frac{1}{2}\right]$ such that

$$\frac{1}{2} + \varepsilon > \frac{\theta}{2-\theta}\left(\frac{1}{2} - \varepsilon\right) \;\Rightarrow\; \varepsilon > \frac{\theta - 1}{2}$$

   So when $\varepsilon > \frac{\theta-1}{2}$, all cars with $v \in \left[p, \frac{1}{2} + \varepsilon\right]$ are not traded. As ε (uncertainty of value) decreases, the equilibrium becomes more efficient as buyers are willing to pay a higher price because the quality of the worst car increased.

3. (MSU Prelim SS 2012 P2 Q2) Suppose there are two types of workers differentiated by productivity level, θ. Good workers produce θ_H while bad workers produce θ_L where $\theta_H > \theta_L > 0$. Prospective employers cannot observe productivity. Suppose that $\Pr(\theta = \theta_H) = 1/2$. Suppose that prior to applying for jobs, workers can accumulate "ratings," $r \in \mathbb{R}_+$. The accumulation of ratings requires effort. In particular, suppose that a worker's preference is given by

$$u(w, r; \theta) = w - \mu r/\theta$$

where w is the wage payment and $\mu > 0$ is a parameter. The labor market is assumed to be competitive, that is, $w = \mathbb{E}(\theta \mid r)$. Answer the following questions. [*Hint:* It might help to approach this problems graphically as done in the class in the context of the Spence model.]

(a) Identify the set of separating (weak)PBE. What is the most efficient separating equilibrium?

Solution.

In a separating wPBE, the firm's offer has the form

$$w(\theta) = \begin{cases} \theta_H & \text{if } r = r_H \\ \theta_L & \text{otherwise} \end{cases}$$

with the belief $P(\theta = \theta_H \mid r = r_H) = 1$. Given that, if an agent does not want to signal that they are θ_H type, then they would pick $r = 0$. To pin down r_H, check incentive compatibility.

For θ_H-type, incentive compatibility holds means:

$$\theta_H - \frac{\mu \cdot r_H}{\theta_H} \geq \theta_L \iff r_H \leq \frac{\theta_H}{\mu}(\theta_H - \theta_L)$$

For θ_L-type, incentive compatibility holds means:

$$\theta_H - \frac{\mu \cdot r_H}{\theta_L} \leq \theta_L \iff r_H \geq \frac{\theta_L}{\mu}(\theta_H - \theta_L)$$

For the most efficient separating equilibrium, r_H must be the exact point where IC binds. As such, the most efficient equilibrium has $r_H = \frac{\theta_L}{\mu}(\theta_H - \theta_L)$.

See part (b) for the graphs.

(b) Identify the set of pooling (weak)PBE. What is the most efficient pooling equilibrium?

Solution.

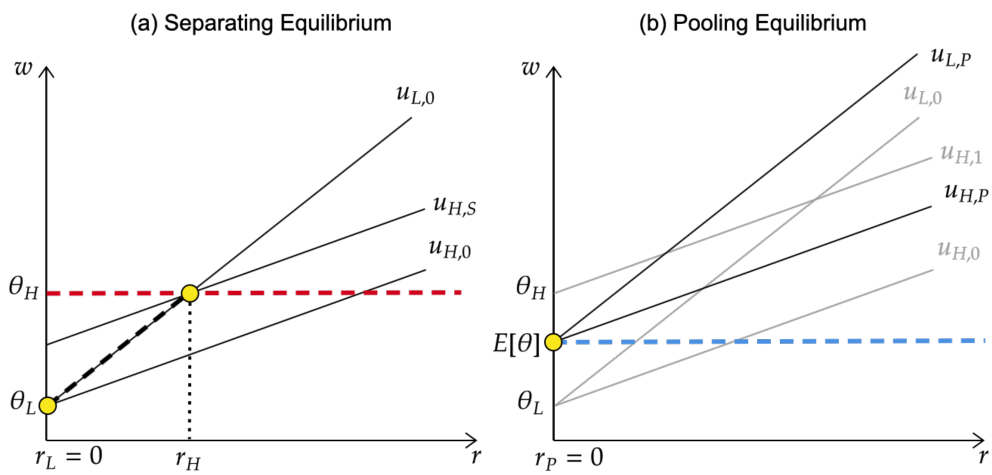
In the set of pooling wPBEs, since the types are not identifiable, the firm will offer

$$w(\theta) = \begin{cases} \frac{1}{2}(\theta_L + \theta_H) & \text{if } r = r_P \\ \theta_L & \text{otherwise} \end{cases}$$

and they have the belief $P(\theta = \theta_H \mid r = r_P) = \frac{1}{2}$. To pin down the most efficient r_P, we can make the incentive compatibility constraint bind:

$$\frac{1}{2}(\theta_H + \theta_L) - \frac{\mu r_P}{\theta_H} = \frac{1}{2}(\theta_H + \theta_L) - \frac{\mu r_P}{\theta_L} \iff r_P = 0$$

So the most efficient pooling equilibrium is $r_P = 0$.



(c) Consider the most efficient separating (weak)PBE. How do the equilibrium payoffs of the two types of the workers change when μ changes?

Solution.

Notice first that in the most efficient separating equilibrium, θ_L-type has $r_L = 0$. As such, their utility is simply θ_L and is thus independent of μ. As for θ_H-type, their payoff under r_H is

$$\theta_H - \frac{\mu}{\theta_H}\underbrace{\frac{\theta_L}{\mu}(\theta_H - \theta_L)}_{r_H} = \theta_H - \theta_L - \frac{\theta_L^2}{\theta_H}$$

which is also independent of μ. As such, equilibrium payoffs of the two types of workers do not change when μ changes.

(d) Now, suppose that an unemployment benefit becomes available that gives a worker a payment of $B \in (\theta_L, \theta_H)$ if he does not accept employment. How will this benefit change the payoffs of the two types of the workers in the most efficient pooling and separating equilibria?

Solution.

For the separating equilibrium, since $B > \theta_L$, the θ_L-type will be unemployed. However, we need to check that it is still incentive compatible for the θ_H-type to separate. The most efficient separating equilibrium thus must satisfy:

$$\theta_H - \frac{\mu}{\theta_H} r'_H = B \iff r'_H = \frac{\theta_H}{\mu}(\theta_H - B) < \frac{\theta_H}{\mu}(\theta_H - \theta_L) = r_H$$

So the rating required for high-type to separate is lowered (because low-type has less incentive to pool). In equilibrium then, both types get higher payoff.

For the pooling equilibrium, since the types are not identifiable, both types would act in the same way: If $B > E[\theta]$, then both types stay unemployed, and if $B \leq E[\theta]$, then both types will be employed with $r \in [0, r^*]$ where

$$B \leq \frac{1}{2}(\theta_H + \theta_L) - \frac{\mu}{\theta_L} r^* < \frac{1}{2}(\theta_H + \theta_L) - \frac{\mu}{\theta_H} r^*$$

4. (MSU 2019 Final Q3) Consider a variation of the Spence signaling model where education is indeed productive. The worker can be one of two types: skilled (S) or unskilled (U). There is a prior probability $\frac{1}{2}$ that the worker is skilled. The worker can choose some education level e. The productivities are $2+e$ if the worker is skilled and $1+e$ if he is unskilled. The cost of education is $\frac{1}{4}e^2$ for a skilled type and $\frac{1}{2}e^2$ for an unskilled type. The worker learns his type, then chooses a level of education, the firm observes this choice and employs the worker at a wage w. The worker's payoff is wage minus the cost of education. The firm faces competition in the labor market and makes zero profit. That is, the worker's wage w is exactly his expected product (given his signal). [Note: This is a slightly generalized version of Spence model discussed in class. The only difference is that here the education is indeed useful in the production process—the worker's productivity increases in e.]

   (a) Suppose there is full information, i.e., the firm can observe the worker's type as well as education level. Find the educational level each type will choose in the SPNE.

   Solution.
   Given that the firm faces competition, they must offer the workers exactly their productivity. Since the signal is productive, there is some optimal e_H and e_L for each type of worker.
   A skilled worker solves:

$$\max_e 2+e-\frac{1}{4}e^2 \;\Rightarrow\; e_H = 2$$

   An unskilled worker solves:

$$\max_e 1+e-\frac{1}{2}e^2 \;\Rightarrow\; e_L = 1$$

   In SPNE, we have

$$w(e) = \begin{cases} 2+e\,, & \text{Worker is skilled} \\ 1+e\,, & \text{Worker is unskilled} \end{cases}\,,\; e(S)=2\,,\; e(U)=1$$

(b) Assume asymmetric information: the firm cannot directly observe the worker's type. Characterize all the pooling weak PBE.

Solution.

Under asymmetric information, the firm cannot distinguish the two types of workers. In a pooling equilibrium, it must be that all workers obtain some education e_p and the firm will pay $\frac{1}{2}(1+e_p)+\frac{1}{2}(2+e_p)=\frac{3}{2}+e_p$.

The skilled workers solve:

$$\max_e \ 1+e-\frac{1}{4}e^2 \ \Rightarrow \ e=2 \ \Rightarrow \ u=3-1=2$$

The unskilled workers solve:

$$\max_e \ 1+e-\frac{1}{2}e^2 \ \Rightarrow \ e=1 \ \Rightarrow \ u=2-\frac{1}{2}=\frac{3}{2}$$

This means that the wage $w_p=w(e_p)$ must satisfy:

$$\begin{cases} \frac{3}{2}+e_p-\frac{1}{4}e_p^2 \geq 2 \\ \frac{3}{2}+e_p-\frac{1}{2}e_p^2 \geq \frac{3}{2} \end{cases}$$

Solving these two inequalities, we get

$$\begin{aligned} &\frac{3}{2}+e_p-\frac{1}{4}e_p^2 \geq 2 \ \Rightarrow \ e_p^2-4e_p+2 \leq 0 \ \Rightarrow \ e_p \in \left[2-\sqrt{2}, 2+\sqrt{2}\right] \\ &\frac{3}{2}+e_p-\frac{1}{2}e_p^2 \geq \frac{3}{2} \ \Rightarrow \ 0 \leq e_p \leq 2 \end{aligned}$$

As such, the pooling wPBEs can be characterized as:

$$w(e)=\begin{cases} \frac{3}{2}+e \ , & e \in \left[2-\sqrt{2}, 2\right] \\ 1+e \ , & \text{Otherwise} \end{cases} \ , \ e(S)=e(U)=e_p \in \left[2-\sqrt{2}, 2\right]$$

$$\mu\left(S \mid e \in \left[2-\sqrt{2}, 2\right]\right)=\frac{1}{2}, \ \mu\left(S \mid e \notin \left[2-\sqrt{2}, 2\right]\right)=0$$

(c) Assume asymmetric information: the firm cannot directly observe the worker's type. Characterize all the separating weak PBE.

Solution.

In a separating equilibrium, skilled workers will choose some e_S and unskilled workers will choose some e_U. Under the single crossing condition satisfied by the cost functions in this question, it must be that $e_S > e_U$.

In equilibrium, skilled workers must not want to deviate. If they deviate, they solve:

$$\max_e 1 + e - \frac{1}{4}e^2 \ \Rightarrow \ e = 2 \ \Rightarrow \ u = 2$$

So it must be that

$$2 + e_S - \frac{1}{4}e_S^2 \geq 2 \ \Rightarrow \ e_S \leq 4$$

Notice that, in a separating equilibrium, the unskilled worker's deviation would be to pretend to be a skilled worker, since otherwise the deviation is not real. As such, for the unskilled workers do not deviate, it must be that

$$1 + e_U - \frac{1}{2}e_U^2 \geq 2 + e_S - \frac{1}{2}e_S^2$$

When the firm can tell that an unskilled worker is unskilled, their optimal e would be the same as e_U, otherwise e_U is not chosen optimally by the unskilled worker. From (b), we know that means $e_U = 1 \ \Rightarrow \ u = \frac{3}{2}$. For the inequality to hold then, it must be that:

$$\frac{3}{2} \geq 2 + e_S - \frac{1}{2}e_S^2 \ \Rightarrow \ e_S \geq 1 + \sqrt{2}$$

As such, the separating wPBEs can be characterized as:

$$w(e) = \begin{cases} 2 + e \ , & e \in \left[1 + \sqrt{2}, 4\right] \\ 1 + e \ , & \text{Otherwise} \end{cases} \ , \ e(S) \in \left[1 + \sqrt{2}, 4\right] \ , \ e(U) = 1$$

$$\mu\left(S \mid e \in \left[1 + \sqrt{2}, 4\right]\right) = 1, \ \mu\left((S \mid e \notin \left[1 + \sqrt{2}, 4\right]\right) = 0$$

5. (BU Prelim 2018) Consider the following two player game. Player 1 is involved in an accident with player 2. Player 1 knows whether he is negligent or not. Player 2 does not know if player 1 is negligent, and assigns probability $\frac{3}{4}$ to him being negligent (these beliefs are common knowledge). If the case goes to court, the judge will learn the truth.

Before the trial, player 1 sends a take-it-or-leave-it settlement offer to player 2, which can be either 3 or 5. Player 2 either accepts or rejects the offer. If he accepts it, the parties don't go to court and the game ends. If he rejects it, the parties go to court and player 1 has to pay 5 to player 2 if he is negligent and 0 if he is not negligent; in either case, player 1 has to pay court expenses of 6.

The payoffs of the game can be epresented by the table:

P_1 \ P_2	Accept	Reject	Accept	Reject
3	$-3, 3$	$-6, 0$	$-3, 3$	$-11, 5$
5	$-5, 5$	$-6, 0$	$-5, 5$	$-11, 5$
	P_1 Not Negligent		P_1 Negligent	

(a) Draw the game tree of this game

(b) Find all the pooling pure strategy wPBE of this game.

Solution.

In the pooling equilibrium, P_2 uses their prior to make their decision. Their expected payoff for rejecting the offer is

$$\frac{3}{4} \cdot 5 + \frac{1}{4} \cdot 0 = \frac{15}{4} \in (3, 5)$$

As such, in a pooling pure strategy wPBE, P_2 will always accept the settlement of 5 and always reject the settlement of 3.

The two pooling pure strategy wPBEs are as follows:

(1) P_1 offers 3 no matter what, P_2 rejects 5 and rejects 3, and the belief is $\mu(\text{negligent} \mid 3) = \frac{3}{4}$, $\mu(\text{negligent} \mid 5) = 1$.

(2) P_1 offers 5 no matter what, P_2 accepts 5 and rejects 3, and the belief is $\mu(\text{negligent} \mid 3) = 1$, $\mu(\text{negligent} \mid 5) = \frac{3}{4}$.

(c) Construct a partially separating wPBE of this game in which P_1 plays a pure strategy when he is not negligent, and plays a mix strategy when he is negligent.

Solution.

Let $\mu(s) = \mu(\text{negligent} \mid s)$ denote P_2's belief when offered settle s. Suppose that P_2 plays the mixed strategy $p(s)A + [1 - p(s)]R$. If P_1 is negligent and willing to mix, it must be that

$$(-3) \cdot p(3) + (-11) \cdot [1 - p(3)] = (-5) \cdot p(5) + (-11) \cdot [1 - p(5)]$$
$$\iff 8p(3) = 6p(5) \iff p(3) = \frac{3}{4}p(5)$$

From (b), we know that accepting $s = 5$ is a weakly dominant strategy for P_2, so it must be that $p(5) = 1$ and $p(3) = \frac{3}{4}$. This means

$$3 = \mu(3) \cdot 5 + [1 - \mu(3)] \cdot 0 \iff \mu(3) = \frac{3}{5}$$

Since $p(5) = 1$ and $p(3) = \frac{3}{4}$, P_1's payoff for offering $s = 3$ when not negligent are:

$$\frac{3}{4} \cdot (-3) + \frac{1}{4} \cdot (-6) = \frac{-15}{4} > -5$$

So P_1 will offer only $s = 3$ when they are not negligent. Given this, when P_1 is negligent, they must mix $q(s = 3) + (1 - q)(s = 5)$ to make $\mu(3) = \frac{3}{5}$ consistent, meaning:

$$\mu(3) = \frac{3}{5} = \frac{\frac{3}{4} \cdot q}{\frac{1}{4} \cdot 1 + \frac{3}{4} \cdot q} \iff q = \frac{1}{2}$$

As such, the partially separating PBE is:

$$s_1^{Neg} = \frac{1}{2}(s = 3) + \frac{1}{2}(s = 5)$$
$$s_1^{NNeg} = (s = 3)$$
$$s_2 = \text{Accept } s = 5 \text{ with prob. } 1 \text{ and Accept } s = 3 \text{ with prob. } \frac{3}{4}$$
$$\mu(s) = \begin{cases} 1 & , s = 5 \\ \frac{3}{5} & , s = 3 \end{cases}$$

6. (MSU Prelim FS 2020 Q3) President Stanley announced a new opt-in option to NS/S grading policy during the coronavirus semester. Let's explore it and an alternative grading policy. Assume an individual's wage-earning ability is perfectly reflected by the grade, which will be a real number uniformly distributed between 0 and 4, $U[0,4]$ (to clarify, any number between 0 and 4—3.45, 1.23, or 0.22—is possible). Suppose a student perfectly observes his grade θ, and the grade is posted in the university system.

   (a) He decides to disclose the exact grade, or pay a small cost $c \in (0, 1/2)$ to opt-in to disclose the grade as Satisfactory (S) if it is between 2 and 4 (inclusive), or to disclose the grade as Non-Satisfactory (NS) if it is between 0 and 2. Upon observing the grade, two firms compete for the student so that the student is paid the competitive wage based on the disclosed grade. Describe the PBE, i.e., student's equilibrium decision to disclose the grade, equilibrium wage, and firms' equilibrium belief.

Solution.

For a student with $\theta \geq 2$, their choice is between disclosing θ (and get wage θ) or only disclose S (and get net wage $\theta_S - c = E[\theta \mid S] - c$). As such, the student will prefer to reveal θ if and only if:

$$\theta \geq \theta_S - c = E[\theta \mid \theta < \theta_S - c] - c = \frac{\theta_S - c + 2}{2} - c$$

A student of type $\theta_S - c$ is thus indifferent, meaning we know θ_S is:

$$\theta_S - c = \frac{\theta_S - c + 2}{2} - c \iff \theta_S = 2 - c < 2$$

So a student will reveal θ if and only if $\theta \geq \theta_S - c = 2 - 2c$. However, since the student is able to reveal S, it must have been that $\theta \geq 2$, so it would not make sense that $\theta_S < 2$. So all students will reveal and an employer will pay 2 for anyone reporting S.

(I will now demonstrate another way to solve this problem, using the second half of this problem)

For a student with $\theta \leq 2$, their choice is between reporting θ or NS. Suppose that there is a point $\underline{\theta}$ where all students with $\theta \in [0, \underline{\theta}]$ would prefer to report

NS, then it must be that:

$$\underline{\theta} = E[\theta \mid NS] - c = E[\theta \mid \theta \leq \underline{\theta}] - c = \frac{0 + \underline{\theta}}{2} - c \iff \underline{\theta} = -2c < 0$$

Since $\theta \geq 0$, a firm would then pay 0 to someone revealing NS.

The PBE is as follows:

For $c > 0$ (given in the question), the student's equilibrium strategy is:

$$s(\theta) = \theta$$

Equilibrium wage offered by the firm is:

$$w = \begin{cases} \theta & \text{if revealed } \theta \\ 2 & \text{if revealed } S \\ 0 & \text{if revealed } NS \end{cases}$$

The firm's belief is:

$$\mu((2 \mid S) = 1$$
$$\mu(0 \mid NS) = 1$$

(b) Suppose, instead of opting in the NS/S option, a student needs to pay a cost to opt *out* of the NS/S option: S is reported for any grade above 2 (inclusive) and NS is reported for any grade below 2 unless he pays a cost $c \in (0, 1/2)$ to report grade θ. Describe the PBE, i.e., equilibrium decision, wage, and belief.

Solution.

For a student with $\theta \geq 2$, their choice is between disclosing θ (and get net wage $\theta - c$) or only disclose S (and get wage $\theta_S = E[\theta \mid S]$). Suppose that there exists $\bar{\theta}$ such that the student reveals if and only if $\theta > \bar{\theta}$, then it must be that

$$\bar{\theta} - c = E[\theta \mid \theta < \bar{\theta}] = \frac{2 + \bar{\theta}}{2} \iff \bar{\theta} = 2 + 2c$$

So those with $\theta > 2 + 2c$ reveals θ and those with $\theta \in [2, 2 + 2c]$ would report S.

Similarly, for a student with $\theta \leq 2$. Suppose that there is a point $\underline{\theta}$ where all students with $\theta \in [0, \underline{\theta}]$ would prefer to report NS, then it must be that:

$$\underline{\theta} - c = E[\theta \mid NS] = E[\theta \mid \theta \leq \underline{\theta}] = \frac{0 + \underline{\theta}}{2} \iff \underline{\theta} = 2c \in (0, 1)$$

So those with $\theta \in [2c, 2)$ would reveal θ and NS for $\theta < 2c$. The PBE is:
For $c > 0$ (given in the question), the student's strategy is:

$$s(\theta) = \begin{cases} \theta & \text{if } \theta \in [2c, 2) \cup [2 + 2c, 4] \\ S & \text{if } \theta \in [2, 2 + 2c) \\ NS & \text{if } \theta \in [0, 2c) \end{cases}$$

Equilibrium wage is:

$$w = \begin{cases} \theta & \text{if revealed } \theta \\ \theta_S = 2 + c & \text{if revealed } S \\ \theta_{NS} = c & \text{if revealed } NS \end{cases}$$

The firm's belief is:

$$\mu((\theta \in [2, 2 + 2c) \mid S) = 1$$
$$\mu(\theta \in [0, 2c) \mid NS) = 1$$

3 Go the Extra Mile

1. (MSU 2020 Final Q2) A medical mask can be highly effective $\theta_H = 1$ or less effective $\theta_L = 0$ against the coronavirus, with an equal probability, $\lambda = \frac{1}{2}$. The general public buying the masks cannot distinguish the effectiveness of the masks. However, the manufacturers making effective mask for their better equipment and materials can more easily produce visibly more sophisticated mask wrappers (that by themselves are completely useless against the coronavirus). The cost of producing the mask wrappers of sophistication level e is given by $\frac{(2-\theta)e^2}{2}$. Two buyers are willing to pay up to θ for a type-θ mask, but they can only distinguish the effectiveness of masks but only the wrappers.

   (a) Describe a pooling PBE. (Hint: State equilibrium effort, price schedule, and belief both on and off equilibrium path. Graphs may help.)

   (b) Describe a separating PBE that does not survive the Intuitive Criterion. (Hint: State equilibrium effort, price schedule, and belief both on and off equilibrium path. Graphs may help.)

   (c) Describe a PBE that survives the Intuitive Criterion.

2. (Columbia Prelim 2011) Consider the following signaling game. Player 2 can be one of two types $t \in \{1, 2\}$, and they know t but P_1 only knows that $P(t = 1) > \frac{1}{2}$. The game under each type with associated payoff matrices is given below:

P_1 \ P_2	L	R	L	R
U	$-3, 3$	$-6, 0$	$-3, 3$	$-11, 5$
D	$-5, 5$	$-6, 0$	$-5, 5$	$-11, 5$
	$t = 1$		$t = 2$	

   Suppose that P_2 can signal their type to P_1 by letting P_1 observe their action before $P - 1$ plays.

   (a) Show that there is a (weak) Perfect Bayesian Equilibrium in which the two types of P_2 pick the same signal. Is there more than one such equilibrium?

   (b) Show that there is a pure-strategy Perfect Bayesian Equilibrium in which the two types of P_2 pick different signals. Is there more than one such equilibrium?

   (c) Derive all mixed-strategy Perfect Bayesian Equilibrium in this game.