[SP26] ECN 812B Recitation 11

Willy Chen

1 Concepts this Week

- **Strict Dominance** : Take some wage schedule set W^*. Effort level e is strictly dominated for type θ if $\exists\, e' \neq e$ such that

$$\inf\left\{W^*(\Theta, e') - c(e', \theta)\right\} > \sup\left\{W^*(\Theta, e) - c(e, \theta)\right\}$$

- Reasonable Type Set: The type set $\Theta^*(e)$ is a reasonable type set if it assigns strictly positive probability to type θ if and only if e is not strictly dominated for type θ.

- **Equilibrium Dominance**: A PBE effort level e is equilibrium dominated for type θ if:

$$u^*(\theta) > \sup\left\{W^*(\Theta, e) - c(e, \theta)\right\}$$

  where $u^*(\theta)$ is the equilibrium payoff of type θ

- A type set $\Theta^{**}(e)$ is **equilibrium reasonable** if

$$\Theta^{**}(e) \equiv \{\theta \mid e \text{ is not equilibrium dominated for } \theta\}$$

- Cho-Kreps: The PBE (σ, μ) <u>violates</u> the **Cho-Kreps Intuitive Criterion** if there exists e' and θ such that

$$u^*(\theta) < \inf\left\{W^*(\Theta^{**}(e'), e') - c(e', \theta)\right\}$$

  Cho-Kreps removes all inefficient separating PBEs as well as all pooling PBEs when the single-crossing condition holds.

  In other words, a PBE violates the Cho-Kreps Intuitive Criterion if there is an off-path deviation that would yield a strictly better payoff for some type , assuming the receiver restricts their beliefs to only include types for whom that deviation is not equilibrium dominated.

- The key difference between equilibrium dominance and strict dominance is the benchmark used for comparison. Strict dominance requires an action to be unconditionally worse than some alternative action, regardless of receiver beliefs. Equilibrium dominance only requires an off-path action to yield a worse payoff than the status quo equilibrium payoff, $u^*(\theta)$, even under the most favorable receiver beliefs. Because comparing a deviation against a guaranteed equilibrium payoff is a weaker condition, it is much easier for an action to be equilibrium dominated than strictly dominated. Consequently, Cho-Kreps can eliminate types that strict dominance cannot.

2 Learning by Doing

1. Consider a game of charitable giving, in which donors care about the esteem that they receive from being considered generous by the public. There are two types of donors, generous (G) and miserly (M), which are equally likely in the population (i.e., the prior belief is $\alpha = P(G) = \frac{1}{2}$). The type of each donor is private information. Each donor chooses an amount x to donate to a charity. The individual donor's utility function is:

$$u_i(\theta_i, x, q) = -(x - \theta_i)^2 + q$$

where $\theta_i(G) = 1$ and $\theta_i(M) = \frac{1}{2}$. The term q is the expected esteem of the donor, where q is the public's equilibrium belief that the donor is generous. The timing of the game is as follows: Each donor chooses how much to donate to the charity $x \in [0, \infty]$, donations become publicly observable, and then the public forms a belief regarding each type of donor, and donors' payoffs are realized.
In what follows, let $x^*(\theta_i), q^*(x^*(\theta_i))$ denote the equilibrium donation by type θ_i and the corresponding equilibrium belief.

(a) Find the range of donations sustainable in a pooling PBE defined by $x^P(G) = x^P(M) = x^P$ and $q^P(x) = 0$ for any $x \neq x^P$.

Solution.
In a pooling equilibrium, both types donate x^P (so the public cannot differentiate, $q(x \neq x^k P) = 0$) and their utility would be $u_i\left(\theta_i, x, \frac{1}{2}\right) = -(x - \theta_i)^2 + \frac{1}{2}$.
By pooling, the miserly type gains $\frac{1}{2}$ (from q), so their IC condition is:

$$-\left(x^P - \frac{1}{2}\right)^2 + \frac{1}{2} \geq 0 \iff x^P \leq \frac{1 + \sqrt{2}}{2}$$

Similarly, the generous type has the IC constraint:

$$-\left(x^P - 1\right)^2 + \frac{1}{2} \geq 0 \iff \frac{2 - \sqrt{2}}{2} \leq x^P \leq \frac{2 + \sqrt{2}}{2}$$

As such, the range of donations in a pooling PBE is $x^P \in \left[\frac{2-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}\right]$

(b) The Cho-Kreps intuitive criterion ensures that in equilibrium, $P(\theta_i \mid x) = 0$ for all x that are equilibrium dominated for type θ_i as long as x is not equilibrium dominated for all types. Recall that x^* is equilibrium dominated for type θ_i given $x^*(\theta_i), q^*(x^*(\theta_i))$ if $u(\theta_i, x^*, q^*) > u(\theta_i, x, q)$ for all q. Show that there is no pooling equilibrium that satisfies the Cho-Kreps intuitive criterion. (Note that the off-equilibrium beliefs may differ from the environment in the previous part.)

Solution.

From (a), we know that the donation ranges for the pooling equilibrium is $\left[\frac{2-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}\right]$. By the definition of Cho-Kreps, we just need to show that for each x^P in this range, there is some x' that one of the types can donate to be better off. Using the Incentive Compatibility condition of the miserly type, if a person makes a donation $x' \in \left(\frac{1+\sqrt{2}}{2}, 2\right]$, the public's equilibrium reasonable type set should be

$$\Theta^{**}(x') = \left\{ \theta = 1 \middle| x' \in \left(\frac{1+\sqrt{2}}{2}, 2\right] \right\} \Rightarrow q(x') = 1$$

Take $x' = \frac{1+\sqrt{2}}{2} + \varepsilon > \frac{1}{2}$ for some small $\varepsilon > 0$. In the pooling equilibrium, the generous type's payoff is $\frac{1}{2} - \left(x^P - 1\right)^2 \leq \frac{1}{2}$. So if the generous type is willing to donate more to distinguish themselves, it must be that:

$$\frac{1}{2} \leq 1 - \left(x' - 1\right)^2 \iff (x'-1)^2 \leq \frac{1}{2} \iff \underbrace{\frac{2-\sqrt{2}}{2}}_{\leq \frac{1}{2}} \leq x' \leq \frac{2+\sqrt{2}}{2}$$

So for any x^P that is a pooling equilibrium, there is always $x' \in \left(\frac{1+\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2}\right]$ that can dominate x^P for the generous type. As such, there is no pooling equilibrium that satisfies the Cho-Kreps Intuitive Criterion.

(c) Specify a fully separating PBE (x^G, x^M) that maximizes the individual expected donations and satisfies the Cho-Kreps intuitive criterion. Carefully specify the belief structure that supports this PBE.

Solution.

In a fully separating PBE, the miserly type does not get any esteem ($q = 0$), as such, by their IR constraints, it must be that $x^M = \frac{1}{2}$. To make the miserly type not want to deviate from $x^M = \frac{1}{2}$, we need the IC constraint to hold for the miserly type, namely:

$$0 \geq \underbrace{1 - \left(x^G - \frac{1}{2}\right)^2}_{\substack{\text{Utility when successfully} \\ \text{pretending to be generous}}} \iff x^G \leq \frac{1-2}{2} < 0 \vee x^G \geq \frac{1+2}{2} = \frac{3}{2}$$

Since donations cannot be negative, it must be that $x^G \geq \frac{3}{2}$ in the separating equilibrium. To pin down the x^G that satisfies Cho-Kreps, x^G must solve:

$$\max_{x \geq \frac{3}{2}} 1 - (x-1)^2 \;\Rightarrow\; x^G = \frac{3}{2}$$

The separating PBE is thus $x^G = \frac{3}{2}$, $x^M = \frac{1}{2}$, and

$$q(\theta = 1 \mid x) = \begin{cases} 1 & \text{if } x = \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$$

This PBE satisfies Cho-Kreps because the miserly type cannot do any better by pretending to be generous, nor can they do any better by donating any $x \neq \frac{1}{2}$. Similarly, the generous type has no incentive to donate more with nothing in return, nor do they have an incentive to donate less, as that would give the miserly type an incentive to pretend to be generous.

(d) What is the individual expected donation under the equilibrium derived in the last part? What is the equilibrium payoff for each type?

Solution.

The individual expected donation is $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{3}{2} = 1$. The equilibrium payoff of miserly type is 0 and of generous type is $\frac{3}{4}$.

Consider now the situation in which the charity employs category reporting instead of revealing the exact donation amount. The silver category includes donations $x < \hat{x}$ and the golden category includes donations that are at least as high as $\hat{x}$. The charity commits to disclosing only the category of the donor, not the actual donation made.

(e) Consider a fully separating PBE in which the generous type donates $x_C^S(G) \geq \hat{x}$ and the miserly type donates $x_C^S(M) < \hat{x}$. For what values of $\hat{x}$ does such an equilibrium exist? How much does each donor donate in this equilibrium? Carefully specify the belief structure that supports this PBE and satisfies the intuitive criterion.

Solution.
For this to be a separating PBE, it must be that the miserly type has no incentive to deviate (IC). From (d), we know that means $\hat{x} > \frac{3}{2}$. It must also be that the generous type has no incentive to not separate (IR), so $\hat{x} \leq 2$.
The PBE is $x^M = \frac{1}{2}$, $x^G \in [\hat{x}, 2]$, $q(G \mid 2 \geq x \geq \hat{x}) = 1$ where $\hat{x} \in \left(\frac{3}{2}, 2\right]$.

(f) What value $\hat{x}$ generates the highest expected individual donation under a fully separating equilibrium such that the generous type donates $x_C^S(G) \geq \hat{x}$ and the miserly type donates $x_C^S(M) < \hat{x}$? Denote the value $\hat{x}^*$. Given $\hat{x}^*$ and the corresponding equilibrium that you found, does category reporting generate higher or lower expected donations compared to the full disclosure separating equilibrium found in the original setting? Are the two types of donors better or worse off under category reporting?

Solution.
Under a fully separating equilibrium, the miserly type only donates $x^M = \frac{1}{2}$. To get the highest expected individual donation, we must then have the highest $\hat{x}$ possible. From (e), we know that means $x^* = 2$, and the total expected donations would be $\frac{5}{4} > 1$. So category reporting can generate higher expected donations than the original setting, under a separating equilibrium. Since the game between the generous type and the non-profit is a zero-sum game, this means the generous type is worse off in the new scheme, while the miserly type is not affected.

2. Consider a used car market where there are three types of sellers: low, middle, and high.

   A low seller values his own car at the reservation price \$2, but it is worth \$4 to a buyer.

   A middle seller values his car at the reservation price \$10, but it is worth \$12 to a buyer.

   A high seller values his car at the reservation price \$11, but it is worth \$14 to a buyer.

   In the population of sellers, $\frac{1}{3}$ are low type, $\frac{1}{3}$ are middle type, and $\frac{1}{3}$ are high type. There are many more buyers than sellers in this market. Buyers cannot distinguish between different types of sellers. All individuals are risk-neutral.

   (a) If buyers competitively announce price bids for cars, then what would be the highest possible equilibrium price?

   Solution.

   Suppose the buyer bids b in equilibrium, it must be that their expected utility from the car is non-negative. In other words,

$$\frac{1}{3}\mathbb{1}\{b \geq 2\} \cdot (4 - b) + \frac{1}{3}\mathbb{1}\{b \geq 10\} \cdot (12 - b) + \frac{1}{3}\mathbb{1}\{b \geq 11\} \cdot (14 - b) \geq 0$$

   First, notice that offering $b \in \{4, 12, 14\}$ is weakly better than offering $b \in \mathbb{R}_+ \setminus \{4, 12, 14\}$.

   If $b = 4$, then the expected payoff of the buyer is $4 - 4 = 0 \geq 0$.

   If $b = 12$, then the expected payoff of the buyer is $\frac{4-12}{3} + \frac{12-12}{3} + \frac{14-12}{3} = -2 < 0$.

   If $b = 14$, then the expected payoff of the buyer is $\frac{(4-14)+(12-14)+(14-14)}{3} = -4 < 0$.

   As such, in equilibrium, the highest possible price is \$4.

(b) Now consider instead a market in which each seller independently announces the price at which he will sell his car. If a buyer is willing to buy it at this price, then a transaction takes place. For this seller-offer game, find a partially separating equilibrium in which each type of seller offers a different price, and each type has a positive probability of trading. Among such separating equilibria, show the equilibrium in which the highest fraction of cars is expected to sell.

Solution.

In the separating equilibrium, the type $t \in \{l, m, h\}$ seller offers s_t. Denote the reservation price of type t seller r_t. It must be that $s_t \geq r_t$. Notice that if, in equilibrium, that buyers buys either s_m or s_h with probability 1, then the type l seller has an incentive to deviate to those offers. As such, it must be that, in equilibrium, the buyers buy type m and type h with probability q_m and q_h.

By the arguments above, we know q_m and q_h must satisfy:

$$\begin{cases} s_l - 2 \geq \max\{q_m(s_m - 2), q_h(s_h - 2)\} \\ q_m(s_m - 10) = q_h(s_h - 10) \end{cases}$$

To maximize profit for the type l sellers, it must be that $s_l = 4$. If $s_h < s_m$, then type h seller is not profit-maximizing. As such, all separating equilibrium where all three types of sellers have different price is:

$$s_l = 4,\ s_m \in [10, 12],\ s_h \in (s_m, 14],\ \mu(t = t' \mid s_{t'}) = 1$$

and buyer buys $s_l = 4$ with probability 1, s_m with probability $q_m = q_h \cdot \frac{s_h - 10}{s_m - 10}$, and s_h with probability $q_h \in \left(0, \frac{s_l - 2}{s_h - 2}\right)$.

For the equilibrium where the most number of types are traded, we need to maximize $q_m + q_h$. To do so, we must make $q_m(s_m - 2) \approx q_h(s_h - 2)$ so that the first condition from above has binding bounds in both cases. To allow for this s_h must be really close to s_m, otherwise the second equality would not hold. Next, notice that the equilibrium s_m and q_m has an inverse relationship in order for the first inequality to be binding. So the equilibrium with the highest $q_m + q_h$ must be the one where $s_m = 11 \lesssim s_h$. This implies that $q_h \lesssim q_m \lesssim \frac{2}{9}$. In this equilibrium, approximately $\frac{1}{3} + \frac{2}{3} \cdot \frac{2}{9} = \frac{13}{27}$ of all cars are traded.

(c) For the seller-offer game that you considered in part (b), now consider equilibria in which all middle and high sellers offer the same price, but low sellers offer a different price. Among such equilibria in which the high and middle types pool together, show the equilibrium in which the highest fraction of cars is expected to sell.

Solution.

Notice that this is essentially already done in part (b). In this equilibrium, $s_l - 2 = q_m(s_m - 2) = q_h(s_h - 2)$ and $q_m = q_h = \frac{2}{9}$.

(d) For each of the three equilibria that you found in parts (a), (b), and (c), identify whether it is (weakly) Pareto-dominated by one or more of the other two equilibria that you found. Justify your answer by appropriately comparing (interim) expected payoffs.

Solution.

In the equilibrium in (a), total consumer surplus is 0 and the producer surplus is $\frac{2}{3}$. In the equilibrium in (b), total consumer surplus is approximately $\frac{8}{9}$ and the producer surplus is approximately $\frac{2}{9}$. In the equilibrium in (c), total consumer surplus is exactly $\frac{8}{9}$ and the producer surplus is exactly $\frac{2}{9}$. As such, (c) $\prec$ (b) $\prec$ (a).

3. (Modified from Stanford Prelim 2003 Q2) A prospective economics PhD student can be of two types, L and H. A PhD program has to give competitive offers to prospective students and pay a type i student π_i such that $\pi_H > \pi_L$. The students know their own types but the program does not. The students can choose to do a pre-Doc program for length p, which costs the student $c_i(p)$. Notice that a pre-Doc experience does not increase the stipend of the student. Assume that $c_H(p) = p^2$ and $c_L(p) = p$. Throughout this question, focus only on pure-strategy equilibria.

   (a) Does the cost functions given satisfy the single crossing property?

   Solution.
   No. $\frac{dc_H(p}{dp} = 2p$ and $\frac{dc_L(p)}{dp} = 1$.
   Single crossing requires that $2p < 1$ which is only true when $p < \frac{1}{2}$.

   (b) Does a separating equilibrium exists in this model?

   Solution.
   Suppose that such equilibrium exists, then it must be that

$$
\begin{aligned}
IC_H: \quad & \pi_H - p_H^2 \geq \pi_L \\
IC_L: \quad & \pi_L \geq \pi_H - p_H
\end{aligned}
$$

   Combining these two inequalities yield:

$$\pi_H - \pi_L \leq p_H \leq (\pi_H - \pi_L)^{\frac{1}{2}}$$

   As such, a separating equilibrium exists if and only if

$$\pi_H - \pi_L \leq 1$$

Suppose now that in addition to the student's education, the programs can also observe how much money d the student spent on partying during their senior year of college, and the student's utility function is $\pi_i - c_i(p) - d$.

(c) Construct a separating equilibrium where the L type student does no pre-Doc and did not spend money on partying. What is the most efficient equilibrium?

Solution.

In this separating equilibrium, it must be that

$$IC_H: \qquad \pi_H - p_H^2 - d \geq \pi_L$$
$$IC_L: \qquad \pi_L \geq \pi_H - p_H - d$$

Combining these inequalities yield:

$$\pi_H - p_H^2 - d \leq p_H \leq (\pi_H - \pi_L - d)^{\frac{1}{2}}$$

So any d such that $\pi_H - \pi_L - d \in [0, 1]$ will allow for some p_H in equilibrium.

For any d, the lower p_H would be more efficient, meaning $p_H = \pi_H - \pi_L - d$. The H type student thus solves:

$$\max_p \ \pi_H - p_H^2 + (p - \pi_H + \pi_L) \equiv \max_p \ \pi_L + p_H + p_H^2 \ \Rightarrow \ p_H = \frac{1}{2}.$$

So the most efficient equilibrium is $p_H = \frac{1}{2}$ and $d = \pi_H - \pi_L - \frac{1}{2}$.

(d) What happens to the payoff of type H as π_H increases? Give an intuition of your finding.

Solution.

From part (c), the equilibrium payoff of type H student is $\pi_L + \frac{1}{4}$, which is invariant to $]pi_H$. The intuition is that H's maximal utility is constrained by IC_L, and therefore is determined by π_L and not π_H.

4. (Modified UPenn Prelim 2016 Q3) Suppose that the payoff to a firm from hiring a worker of type θ with education e at wage w is

$$f(e,\theta) - w = 3e\theta - w.$$

The utility of a worker of type θ with education e receiving a wage w is

$$w - c(e,\theta) = w - \frac{e^3}{\theta}.$$

The worker's ability is privately known by the worker. There are at least two firms. The worker (knowing their ability) first chooses an education level $e \in \mathbb{R}_+$; firms then compete for the worker by simultaneously announcing wage; finally, the worker chooses a firm. Treat the wage determination as in class, a function $w : \mathbb{R}_+ \to \mathbb{R}_+$ determining wage as a function of education.
Suppose the support of the firms' prior beliefs ρ on $\Theta = \{\theta_L, \theta_H\}$ where $\theta_L = 1$ and $\theta_H = 3$.

(a) Suppose that the firms can perfectly observe the worker's type. What is the most efficient level of education for each type of worker?
Solution.
The social surplus from a worker of type θ taking education level e is:

$$3e\theta - \frac{e^3}{\theta}$$

The F.O.C. is:

$$3\theta - \frac{3e^2}{\theta} = 0 \;\Rightarrow\; e^* = \theta.$$

Low type chooses $e_L = 1$ and high type chooses $e_H = 3$.

(b) Is there a perfect Bayesian equilibrium in which both types of worker choose their full information eduction level? Be sure to verify that all the incentive constraints are satisfied.

Solution.

If each type of worker chooses their respective optimal education level as solved in part (a), we have a separating equilibrium. Since the firms can perfectly infer the worker's type, they have the wage schedule:

$$w(e) = \begin{cases} 3e & \text{if } e \neq 3 \\ 9e & \text{if } e = e_H = 3 \end{cases}$$

For this wage schedule to be part of the equilibrium, it must be that

$$IC_H: \qquad 9e_H - \frac{e_H^3}{3} \geq \max_e \left\{ 3e - \frac{e^3}{3} \right\}$$

$$IC_L: \qquad 3e_L - e_L^3 \geq \max \left\{ \max_e \{3e - e^3\},\, 9e_H - e_H^3 \right\}$$

According to part (a), IC_H holds automatically and that IC_L can be reduced down to

$$3e_L - e_L^3 \geq 9e_H - e_H^3 \;\Rightarrow\; 3 - 1^3 \geq 27 - 3^3$$

which holds. As such, a separating PBE where the two types choose their first-best education does exists.

(c) What is the lowest possible $\rho = P(\theta = \theta_H)$ for there to exist a pooling equilibrium where both types of workers choose the same education.

Solution.

Suppose a pooling equilibrium exists, then both types of workers choose some education level e_p and the firm's wage schedule is:

$$w(e) = \begin{cases} 3e & \text{if } e \neq e_p \\ 3e_p(1+2\rho) & \text{if } e = e_p \end{cases}$$

For this wage schedule to be part of the equilibrium, it must be that

$$IC_H : \quad 3e_p(1+2\rho) - \frac{e_p^3}{3} \geq \max_e \left\{3e - \frac{e^3}{3}\right\} = 2\sqrt{3}$$
$$IC_L : \quad 3e_p(1+2\rho) - e_p^3 \geq \max_e \{3e - e^3\} = 2$$

Simplifying these two conditions we get:

$$\rho \geq \frac{\sqrt{3}}{3e_p} + \frac{e_p^2}{18} - \frac{1}{2} \text{ and } \rho \geq \frac{1}{3e_p} + \frac{e_p^2}{6} - \frac{1}{2}$$

The derivatives of these lower bounds on ρ are:

$$\underbrace{\frac{e_p}{9} - \frac{\sqrt{3}}{3e_p^2}}_{=0 \text{ when } e_p = \sqrt{3}} \quad \text{and} \quad \underbrace{\frac{e_p}{3} - \frac{1}{3e_p^2}}_{=0 \text{ when } e_p = 1}$$

The second derivatives of these lower bounds on ρ are:

$$\frac{1}{9} + \frac{2\sqrt{3}}{3e_p^3} > 0 \text{ and } \frac{1}{3} + \frac{2}{3e_p^3} > 0$$

If $e_p = \sqrt{3}$, the maximum of the lower bounds is $\frac{\sqrt{3}}{9} \approx \frac{1.7}{9}$. If $e_p = 1$, the maximum of the lower bounds is $\frac{\sqrt{3}}{3} + \frac{1}{18} - \frac{1}{2} \approx \frac{1.19}{9}$. So the lowest possible ρ is $\frac{\sqrt{3}}{3} + \frac{1}{18} - \frac{1}{2}$.

3 Go the Extra Mile

1. (Modified from NYU Prelim 2014 Q4) There is one seller with a car for sale, with quality $q \sim U[0,1]$. There is one potential buyer. If a car of quality q is sold at price p, then the seller's payoff is p and the buyer's payoff is $v(q) - p$ where $v(\cdot)$ is a known non-negative strictly increasing function. If the seller keeps the car, then the seller's payoff is q, and the buyer's payoff is 0.

   (a) Suppose the seller announces a price and the buyer either accepts or rejects, what is the equilibrium price?

   Solution.
   If a seller announces price p, then it must be that $q \leq p$. This means the buyer's expected utility is

   $$E[v(q) \mid q \leq p] = \int_0^p v(q) \frac{1}{p} dq$$

   (b) Suppose that the seller can purchase a signal $s \in \mathbb{R}^+$ before announcing a sale price: signal s costs $\frac{s}{q}$ for a seller with a car of quality q. After seeing both the price and the signal, the buyer can either accept or reject the price. Assume that $v(q) = q + b$, $b > 0$. Does there exists a separating PBE where all cars with quality q are sold at $v(q)$. Does your answer depend on b?

2. (Penn Prelim FS 2017 Q3) An entrepreneur is contemplating selling all or part of his startup to outside investors. The profits from the startup are risky and the entrepreneur is risk averse. The entrepreneur's preferences over $x \in [0,1]$, the fraction of the startup the entrepreneur retains, and p, the price "per share" paid by the outside investors, are given by

   $$u(x, \theta, p) = \theta x - x^2 + p(1 - x)$$

   where $\theta > 1$ is the value of the startup (i.e., expected profits). The quadratic term reflects the entrepreneur's risk aversion. The outside investors are risk neutral, and so the payoff to an outside investor of paying p per share for $1 - x$ of the startup is then

   $$\theta(1 - x) - p(1 - x)$$

There are at least two outside investors, and the price is determined by a first price sealed bid auction: The entrepreneur first chooses the fraction of the firm to sell, $1-x$; the outside investors then bid, with the $1-x$ fraction going to the highest bidder (ties are broken with a coin flip). Important convention: The outside investors submit bids in "price per share" p , so the amount paid is $p(1-x)$.

(a) Suppose θ is public information. What fraction of the startup will the entrepreneur sell, and how much will he receive for it?

(b) Suppose now θ is privately known to the entrepreneur. The outside investors have common beliefs, assigning probability $\alpha \in (0,1)$ to $\theta = \theta_1 > 1$ and probability $1-\alpha$ to $\theta = \theta_2 > \theta_1$. Suppose $\theta_2 - \theta_1 > 2$. Characterize the separating perfect Bayesian equilibria. Are there any other perfect Bayesian equilibria?