

[SP25] ECN 812B Recitation 12

Willy Chen

1 Concepts this Week

- Individual Rationality: Every participant's utility is at least as good as their outside option \bar{u} (typically normalized to 0)
- Incentive Compatibility: Every participant's utility of truthfully revealing their type is at least as good as misreporting their type
- Screening
 - In signaling, the agents pay the cost of signal to separate themselves. In screening, the principal pays the agents to self-separate.
 - First-Best: The solutions when there is no information asymmetry.
 - Second-Best: The most efficient contract under information asymmetry.
- Differences between concepts:
 - Adverse Selection:
 - Signaling:
 - Monopolistic Screening:
 - Moral Hazard (next week):

2 Learning by Doing

1. (MSU FS2020 Q3) A third-year college athlete decides whether to forgo her final year of the four-year college eligibility to become a professional athlete. Her payoff is the sum of her athletic earnings and her after-sports earnings. She knows her own true athletic ability $\theta \in [0, 1]$, but her future athletic employer will not know her true ability if she turns pro early. If she becomes a professional athlete early, her athletic earning is (endogenously) determined by the average ability of all those who turn pro early. If she waits one more year, her true ability is fully revealed and her athletic earning equals her athletic ability. The athletic abilities of all college athletes are public known to be uniformly distributed between 0 and 1. Besides the athletic earnings, she will be paid based on her education level after she retires from sports: She will receive wage earnings w_L if she turns pro early, and wage earnings $w_H > w_L$ if she stays fourth year and receives a college degree. If she waits one more year, her athletic earnings and wage earnings are discounted by $\delta \in (\frac{1}{2}, 1]$. In summary, her payoff is $\delta(v + w_H)$ if she stays one more year, and is $v + w_L$ if she turns pro early, where v is her equilibrium athletic earnings.

- (a) Describe the competitive equilibrium under different parameters, specifying her equilibrium decision and equilibrium wage if she turns pro early.

Solution.

In equilibrium, a type θ athlete waits one more year if and only if:

$$\delta(\theta + w_H) \geq v_L + w_L \iff \theta \geq \frac{v_L + w_L}{\delta} - w_H$$

On the margin, $v_L = E[\theta \mid \theta < \theta^*]$, so we can solve for θ^* recursively:

$$\delta\theta^* = \overbrace{E[\theta \mid \theta < \theta^*]}^{\theta^*/2} + w_L - \delta w_H \Rightarrow \theta^* = \frac{w_L - \delta w_H}{\delta - \frac{1}{2}}$$

then all athletes with $\theta \geq \theta^*$ will wait. So the wage of those not waiting is $E[\theta \mid \theta \leq \theta^*] = \frac{\theta^*}{2} = \frac{w_L - \delta w_L}{2\delta - 1}$.

(b) Suppose an athlete can pay an extra cost $c > 0$ (e.g., hire an agent, showcase her ability in pre-draft trials) to reveal her true ability. Assume $w_L > \delta w_H$. Describe a competitive equilibrium in which all three decisions (i.e., turning pro early without paying, turning pro with a cost, or waiting) are present in equilibrium, specifying her equilibrium decision and equilibrium wage if she turns pro early without paying the extra cost. Characterize the condition that such an equilibrium exists.

Solution.

Since the athlete can also just wait a year to reveal their θ , they choose between:

- Enter early: $v_L + w_L$
- Enter early but pay for agent: $\theta - c + w_L$
- Wait a year: $\delta(\theta + w_H)$

Enter early with agent is better than waiting iff:

$$\theta - c + w_L \geq \delta(\theta + w_H) \Rightarrow \theta \geq \frac{c - (w_L - \delta w_H)}{(1 - \delta)} = \theta^{**}$$

For $\theta^{**} > 0$, it must be that $c > w_L - \delta w_H$.

Enter early with agent is better than no agent iff:

$$\theta - c > v_L \Rightarrow \theta > v_L + c$$

From (a), entering early without agent is better than waiting iff:

$$\theta < \underbrace{\frac{w_L - \delta w_H}{\delta - \frac{1}{2}}}_{>0}$$

So for higher θ , entering early is likely worse than waiting. So an equilibrium where all three types exists must mean $\theta^{**} > \theta^*$, meaning:

$$\begin{aligned} \frac{c - (w_L - \delta w_H)}{(1 - \delta)} &> \frac{w_L - \delta w_H}{\delta - \frac{1}{2}} \\ \Rightarrow c(2\delta - 1) - (w_L - \delta w_H)(2\delta - 1) &> (2 - 2\delta)(w_L - \delta w_H) \\ \Rightarrow c(2\delta - 1) &> w_L - \delta w_H \Rightarrow c > \frac{w_L - \delta w_H}{2\delta - 1} \end{aligned}$$

As such, $\theta < \theta^*$ enters without agent, $\theta \in [\theta^*, \theta^{**})$ waits, and $\theta \in [\theta^{**}, 1]$ enters with agent, with $w_L > \delta w_H$ and $c > \frac{w_L - \delta w_H}{2\delta - 1}$.

2. (MSU 2021 Final Q3) An agent privately observes her income 0 (low) or 1 (high). It is common knowledge the income is either 0 or 1 with equal probability. The principal (IRS) decides the income tax rate $t \in [0, 1]$. A high-income agent can misreport her income to be low; while the low-income agent can only tell the truth. To prevent the high-income agent's misreporting, the principal audits the agent's income with probability $p \in [0, 1]$ when it is reported to be low. Assume that auditing costs $c \in (0, 1)$. When the agent is caught for misreporting, her income will be taken away. Suppose the agent is risk averse and has the utility function $u(x) = \sqrt{x}$ and the principal is risk neutral.

(a) Suppose that the principal can commit to a policy (t, p) to maximize its expected payoff. Formulate and solve the principal's problem.

Solution.

The IRS solves the problem:

$$\max_{t,p} \frac{1}{2}t + \frac{1}{2}(-pc) \text{ subject to } \sqrt{1-t} \geq 1 - p(IC_H)$$

IC_L is irrelevant since the low type does not have multiple strategies. At optimum, IC_H must bind, meaning $p = 1 - \sqrt{1-t}$. This means the IRS's problem is:

$$\max_t \frac{1}{2}t + \frac{1}{2}(-1 + \sqrt{1-t})c \Rightarrow F.O.C. : \frac{1}{2} + \frac{c}{4\sqrt{1-t}}$$

Solving from the F.O.C. we get:

$$t = 1 - \frac{c^2}{4} \text{ and } p = 1 - \frac{c}{2}.$$

(b) Is the ex ante optimal p you solved in (a) sequentially optimal? Why or why not, explain your reasoning.

Solution.

No. If everyone truthfully reports income in equilibrium, auditing is not sequentially rational since it costs money.

- (c) From now on, assume that $t = 0.5$, $c = 0.2$ and suppose that the principal cannot commit to any auditing strategy ex ante. He will make such a choice after receiving the agent's income report. Solve for the mixed-strategy perfect Bayesian equilibrium where the high type agent reports each type of income with strictly positive probability.

Solution.

Suppose that in such an equilibrium, the high type agent reports low income with probability q . IRS's belief about the agent's type is:

$$P(y = 1 \mid \hat{y} = 0) = \frac{\frac{1}{2} \cdot q}{\frac{1}{2} \cdot q + \frac{1}{2}} = \frac{q}{q + 1}.$$

To make the high type agent indifferent, the IRS must audit with probability p such that:

$$\sqrt{\frac{1}{2}} = \sqrt{1} \cdot (1 - p) \Rightarrow p^* = 1 - \frac{\sqrt{2}}{2}$$

The IRS's expected revenue from auditing with probability p^* is:

$$(1 - c) \cdot \frac{q}{q + 1} + (-c) \cdot \frac{1}{q + 1} = \frac{q}{q + 1} - 0.2$$

For the IRS to be willing to not always audit, it must be that:

$$\frac{q}{q + 1} - 0.2 = 0 \Rightarrow q = \frac{1}{4}$$

As such, in this PBE, the IRS will audit with probability $p^* = 1 - \frac{\sqrt{2}}{2}$. The high type agent will report low with probability $q = \frac{1}{4}$. The low type agent will report low with probability 1. IRS has the belief $P(y = 1 \mid \hat{y} = 0) = \frac{1}{5}$.

3. (MSU 2019 Final Q4) Suppose that a monopolist seller wants to sell a unit of a good to a buyer. The buyer's valuation of good is her private information but it is commonly known that the buyer can have one of two valuations: θ_l (with probability p_l) or θ_H (with probability p_h), where $0 \leq \theta_l < \theta_h$. While the seller is risk neutral, assume that the buyer is risk averse. Moreover, suppose that the seller uses a lottery to sell the good where the lottery is defined as follows: Trade takes place if the buyer wins the lottery. The lottery specifies a winning probability q for the buyer, a price of the good W if the buyer wins, and a penalty L if the buyer loses. A bidder has utility $u(\theta - W)$ when winning and paying W , and $u(-L)$ when losing and paying L . The utility function is strictly increasing and concave. Let the reservation utility of the buyer be $u(0)$.

Now assume that the seller offers a menu of lotteries $\{(q_l, W_l, L_l), (q_h, W_h, L_h)\}$ to the buyer where (q_i, W_i, L_i) is the lottery intended for the buyer of type θ_i , $i \in \{l, h\}$. Assume the production cost is zero and the seller's reservation price is zero. Furthermore, assume $p_l q_l + p_h q_h \leq \frac{1}{2}$ and $q_h \leq p_l + \frac{p_h}{2}$. The optimal screening problem for the seller can now be written as:

$$\max_{\{(q_l, W_l, L_l), (q_h, W_h, L_h)\}} p_l [q_l W_l + (1 - q_l) L_l] + p_h [q_h W_h + (1 - q_h) L_h]$$

subject to

$$q_l u(\theta_l - W_l) + (1 - q_l) u(-L_l) \geq u(0) \quad (IR_L)$$

$$q_h u(\theta_h - W_h) + (1 - q_h) u(-L_h) \geq u(0) \quad (IR_H)$$

$$q_l u(\theta_l - W_l) + (1 - q_l) u(-L_l) \geq q_h u(\theta_l - W_h) + (1 - q_h) u(-L_h) \quad (IC_L)$$

$$q_h u(\theta_h - W_h) + (1 - q_h) u(-L_h) \geq q_l u(\theta_h - W_l) + (1 - q_l) u(-L_l) \quad (IC_H)$$

The following questions asks you to characterize the optimal screening contract.

- (a) Argue that that in the reduced program, the low type's IR constraint and the high type's IC constraint must bind.

Solution.

If IR_L does not bind, the seller can increase W_l a little bit without violating IC_H . If IC_H is not binding, the seller can increase W_h a little bit without violating IR_L .

- (b) Show that in the reduced program the high type is fully insured (i.e., we must have $-L_h = \theta_h - W_h$)

Solution.

Let λ_1 and λ_2 be the Lagrangian multipliers for IR_L and IC_H . The F.O.C.s are

$$\begin{aligned} [W_h] : q_h &= \lambda_2 u'(\theta_h - W_h) \\ [L_h] : q_h &= \lambda_2 u'(-L_h) \\ &\Rightarrow u'(\theta_h - W_h) = u'(-L_h) \end{aligned}$$

Since $u(\cdot)$ is strictly concave, it must be that $L_h = W_h - \theta_h$.

- (c) Show that in the reduced program the high type's IR constraint is slack (strict inequality holds) if and only if $L_h < 0$

Solution.

IR_H is slack if and only if

$$q_h u(\theta_h - W_h) + (1 - q_h)u(-L_h) > u(0) \iff u(-L_h) > u(0) \iff L_h < 0$$

(d) Show that in the reduced program we must have $L_l > 0$

Solution.

Suppose otherwise that $L_l \leq 0$. Notice that $L_l = 0$ would mean $\theta_l - W_l = 0$, because the F.O.C. with respect to W_l and L_l imply

$$\begin{aligned} p_h &= -\lambda_1 u'(\theta_l - W_l) + \lambda_2 u'(\theta_h - W_l) \\ p_h &= -\lambda_1 u'(-L_l) + \lambda_2 u'(-L_l) \end{aligned}$$

So $\theta_l - W_l = L_l = 0$ means $u'(\theta_h - W_l) = u'(0)$.

As $u'' < 0$, this is the case only if $\theta_h - W_l = 0$. But $\theta_h - W_l > \theta_l - W_l = 0$, which is a contradiction.

Now suppose $L_l < 0$. Then $u(-L_l) > u(0)$ and IR_L binds, meaning $u(\theta_l - W_l) \leq u(0)$.

Consider the seller increases L_l slightly and decreases W_l slightly such that the low type's expected utility does not change. Since the consumer receives more insurance, this means the seller can make more revenue. Lastly, this change will make the high type's IC constraint more slack because $u'(\theta_h - W_l) < u'(\theta_l - W_l)$

4. (Modified MSU FS 2018 Q5) A principal needs to buy the services of a skilled artisan and needs to design an optimal contract when the artisan's cost of supplying his services is not known by the principal. If the artisan's cost type is t , then his net utility gain from providing q hours of service from either type would be $u(w, q|t) = w - tq$. The principal's net benefit from paying w for q hours of service from the wage can depend on the artisan's cost type (which the artisan knows), but the artisan could misrepresent his type. The artisan will not work until he gets a non-negative utility gain.

(a) Suppose that the artisan's cost type is either 3 or 5, and the principal thinks that the probabilities are $\Pr(t = 3) = 2/3$ and $\Pr(t = 5) = 1/3$. Find the contract that maximizes the principal's expected net benefit from her business with the artisan, where the principal's gain from trade is $6\sqrt{q} - w$. What is the optimal menu of contracts $\{(w(t), q(t))\}$?

Solution.

Case (i) Suppose the principal offers a contract that simply makes IRL bind. The wage has to be $w \geq 5q$ for both types of workers willing to work. To maximize the principal's profit, he shall make IR bind for the type with $t = 5$. As such, he chooses q to maximize $6\sqrt{q} - 5q$, so $q = 9/25$. The expected profit for the principal is thus $6 \cdot \frac{15}{25} - 5 \cdot \frac{9}{25} = \frac{9}{5}$.

Case (ii) Suppose the principal offers a contract that simply makes IRH bind. Then the wage is $w = 3q$. To maximize profit, he chooses q to maximize $6\sqrt{q} - 3q$, so $q = 1$. The expected profit is $(6 - 3) \cdot \frac{2}{3} = 2 > \frac{9}{5}$.

Case (iii) A contract that binds IR_L, IR_H , and IC_H . The principal's problem is:

$$\begin{aligned} & \max_{q(3), q(5), w(3), w(5)} \frac{2}{3} (6\sqrt{q(3)} - w(3)) + \frac{1}{3} (6\sqrt{q(5)} - w(5)) \\ \text{subject to } & w(3) - 3q(3) \geq w(5) - 3q(5), w(3) - 3q(3) \geq 0 \\ & w(5) - 5q(5) \geq w(3) - 5q(3), w(5) - 5q(5) \geq 0 \end{aligned}$$

Letting IRL and ICH bind, we have $w(5) = 5q(5)$ and $w(3) = 2q(5) + 3q(3)$.

The reduced program is thus:

$$\max_{q(3), q(5)} \frac{2}{3} (6\sqrt{q(3)} - 2q(5) - 3q(3)) + \frac{1}{3} (6\sqrt{q(5)} - 5q(5))$$

Solving this problem we get $q(5) = \frac{1}{9}$, $q(3) = 1$ and thus $w(5) = \frac{5}{9}$, $w(3) = \frac{29}{9}$.

Suppose now it is common knowledge that the artisan's cost type t is drawn from $U[3, 5]$, so that the principal's objective is to choose $(w(t), q(t))$ such that it maximizes

$$\int_3^5 \frac{6\sqrt{q(t)} - w(t)}{2} dt$$

(b) Argue that if the menu $\{(w(t), q(t))\}$ is incentive compatible, then

$$t \in \underset{\tau \in [3,5]}{\operatorname{argmax}} w(\tau) - tq(\tau)$$

and use this fact to show that the optimal contract is such that

$$w(t) = u(w(5), q(5) \mid t = 5) + tq(t) + \int_t^5 q(s) ds$$

Solution.

Incentive compatibility means that for all types t , their payoff for choosing the contract $(w(t), q(t))$ is weakly higher than choosing any other contracts $(w(t'), q(t'))$. As such, it must be the case that

$$t \in \underset{\tau \in [3,5]}{\operatorname{argmax}} w(\tau) - tq(\tau)$$

This maximizing conditions means that we have the F.O.C.

$$w'(\tau) = tq'(\tau)$$

Note that t is the true type and τ is the **reported** type, and IC implies $\tau = t$. By the fundamental theorem of calculus, we can write:

$$\begin{aligned} w(t) - w(5) &= \int_5^t w'(s) ds = \int_5^t sq'(s) ds \\ \Rightarrow w(t) &= \underbrace{w(5)}_{=u(w(5), q(5) \mid t=5) + 5q(5)} - \int_t^5 sq'(s) ds = w(5) - sq(s) \Big|_t^5 + \int_t^5 q(s) ds \\ \Rightarrow w(t) &= u(w(5), q(5) \mid t = 5) + tq(t) + \int_t^5 q(s) ds \end{aligned}$$

- (c) Using the formula for $w(t)$, argue whether $t = 5$ artisan should get any information rent.

Solution.

In the optimal contract from (b), we already assumed that Individual rationality must bind for $t = 5$. As such, under the optimal contract, the $t = 5$ artisan does not get any information rent with $\varepsilon(5) = 0$.

- (d) Given the above formula for $w(t)$, the principal's payoff can be written as:

$$\int_3^5 \frac{6\sqrt{q(t)} - (2t - 3)q(t)}{2} dt$$

Derive the formula for the optimal $q(t)$. What is $q(3)$ under this formula? What can you say about distortion for the $t = 3$ type artisan under this optimal contract?

Solution.

Given this expected payoff, the principal solves:

$$\max_{q(t)} \int_3^5 \frac{6\sqrt{q(t)} - (2t - 3)q(t)}{2} dt$$

Given any $t \in [3, 5]$, we can simplify this maximization problem with point-wise maximization:

$$\max_{q(t)} 6\sqrt{q(t)} - (2t - 3)q(t) \Rightarrow \frac{3}{\sqrt{q(t)}} = 2t - 3 \Rightarrow q(t) = \left(\frac{3}{2t - 3}\right)^2 \Rightarrow q(3) = 1$$

Under this optimal contract, the $t = 3$ artisan would work $q = 1$, which is the same as part (a). This means that there is no distortion at the top and the optimal contract achieves first best.