[SP26] ECN 812B Recitation 13

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1 Concepts this Week

Moral Hazard/**Hidden Action**/Principal-Agent Problem

- In screening (**hidden types)**, the principal pays the cost for agents to self-separate. In moral hazard, the principal doesn't need to pay the cost for agents to self-separate because there is no types. However, there are different levels of efforts that the agent can exert, and the principal's goal is to incentivize the agents to exert the effort that the principal wants.
- The principal *contract* on observables, whether that is outcome, signal, or something else.
  - An object is said to be *contractible* if the payoff can be written as a function of that object.
  - For example, in a course, we contract your final grade (analogous to wage) on your over exam grade (observable signal of your effort).
- Individual Rationality: Every participant's utility is at least as good as their outside option $\bar{u}$ (typically normalized to 0).
- Incentive Compatibility: Every participant's **expected utility** of exerting high effort is at least as good as that of exerting low effort.
- If the agent is risk-neutral, then first-best is achievable
- If the agent is risk-averse, then first-best is not achievable as the firm needs to take on more risk to incentivize/implement a certain effort level.

Final exam is 7:45-9:45 am on May 1st in regular classroom. Good luck on your finals/prelims!

2 Learning by Doing

1. (MSU Final 2020) Consider the following moral hazard problem. There is a firm and a single worker who can either exert effort (e = 1) or shirk (e = 0). If the worker exerts effort, the output is H = 10. If the worker shirks then the output, π, is 10 with probability p and $L \geq 0$ with probability $1-p$ such that $p \cdot 10 + (1-p) \cdot L = 5$. The distribution of output conditional on the worker's effort choice is common knowledge. First, the owner of the firm offers a wage schedule $(w(H), w(L))$ such that $w(\pi) \geq 0$ for both $\pi = H, L$. Then the worker decides whether to exert effort or shirk. If the realized output is π, the owner's payoff is $\pi - w(\pi)$ and the worker's payoff is $w(\pi) - 3e$. Both the owner and the worker are expected payoff maximizers.

   (a) Show that if the owner wants to implement $e = 1$, the worker's incentive constraint binds.

   Solution.
   If the owner wants to implement $e = 1$, then the incentive compatibility constraint is:

   $$w(10) - 3 \geq p \cdot w(10) + (1-p) \cdot w(L)$$
   $$\Rightarrow \; (1-p) \cdot (w(10) - w(L)) \geq 3.$$

   If the owner wants to implement $e = 1$, it must be that:

   $$10 - w(10) \geq 5 - p \cdot w(10) - (1-p) \cdot w(L)$$
   $$\Rightarrow \; 5 \geq (1-p)(w(10) - w(L)).$$

   The owner can thus profit maximize by decreasing $w(L)$ until $(1-p) \cdot (w(10) - w(L)) = 3$, making IC binding.

(b) Show that if the owner wants to implement $e = 1$, then $w(L) = 0$. Derive $w(H)$ in this case.

Solution.

If the owner wants to implement $e = 1$, they can decrease $w(L)$ arbitrarily as long as IR and IC both hold. From (a), we know that IC must bind, so the $w(H) = \frac{3}{1-p}$.

(c) For what values of (p, L) does the owner want to implement $e = 1$?

Solution.

If the owner implements $e = 0$, their payoff is $5 - pw(10) - (1-p)w(L)$. They can maximize this by setting $w(10) = w(L) = 0$ and get the payoff of 5. So if they want to implement $e = 1$, it must be that:

$$10 - w(10) \geq 5 \;\Rightarrow\; 10 - \frac{3}{1-p} \geq 5 \;\Rightarrow\; p \leq \frac{2}{5}$$

(d) Suppose that prior to interacting with the owner, the worker can choose any pair (p, L) as long as $p \cdot 10 + (1-p) \cdot L = 5$. This choice is observed by the owner before he determines the wage schedule. What is the equilibrium choice of the worker?

Solution.

If the worker chooses $p > \frac{2}{5}$, the owner will offer a contract that will yield a realized wage of 0 to the worker and implement effort level 0. If the worker chooses $p \leq \frac{2}{5}$, the owner will offer the optimal contract with $w(L) = 0$ and $w(H) = \frac{3}{1-p}$, and the worker gets a payoff of $\frac{3p}{1-p}$. The payoff is increasing in pp, so the worker will choose $p = \frac{2}{5}$ and $L = \frac{5-10p}{1-p} = \frac{1}{1-2/5} = \frac{5}{3}$.

2. (MSU Prelim FS 2015 P2) A principal P hires an agent A to work on a consulting project. The outcome of the project is $Y \in \{0, y\}$; think of y (> 0) as "success" and 0 as "failure". Y is publicly observed and contractible. A is required to exert costly effort $e \in \{0, 1\}$ to first learn the "state of the world" $\theta \in [0, 1]$ and then perform a task $t \in [0, 1]$. But A can always choose a t even without knowing θ. Cost of effort $e = 1$ is c $(< y)$ and the cost of $e = 0$ is 0. Also, all tasks can be performed at zero cost.

   The project is surely successful, i.e., $Y = y$ if $t = \theta$. If A performs any task $t \neq \theta$, $Y = 0$ except for the following case: there exists a task $t^* \in [0, 1]$ such that if $t = t^*$, for any θ, $Y = y$ with probability p and $Y = 0$ with probability $1 - p$. The agent, however, does not know which task is t^* but P has this information. Note that P does not observe e, the task chosen by A or θ.

   To understand this setting, consider the following scenario: think of θ as the source of the problem that the client is facing. A must exert effort to learn the nature of the problem and then suggest the appropriate strategy to address the concern. Once the problem source is identified, it can surely be fixed by choosing the appropriate strategy. P observes if the client's problem has been solved but does not know what the source was or what strategy A has recommended to address the issue. However, P knows that one of the available strategies is "special" in the sense that irrespective of the source of the client's problem, if this strategy is prescribed, there is some chance that it would resolve the problem. But A does not know which one of the all available strategies is the "special"one.

   P offers a salary $s \geq 0$ and bonus reward $b \geq 0$ if $Y = y$. That is, A's wage contract is of the form: $w(0) = s$ and $w(y) = s + b$. P's payoff is $Y - w(Y)$ and A's payoff is $w(Y) - ce$. The outside options for both P and A is 0.

(a) Suppose P wants to implement $e = 1$ by choosing s and b. Write down P's optimal contracting problem. Clearly specify P's objective function and **explain** the associated (IC) and (IR) constraints (recall that $s, b \geq 0$ must also be included in the set of constraints). (You must explain your derivation of the (IC) to be consider for any credit in this question).

Solution.

P's problem is:

$$\max_{s,b} \quad y - s - b \qquad s.t.$$

$$b - c \geq 0 \qquad (IC)$$

$$s + b - c \geq 0 \qquad (IR)$$

$$s \geq 0, b \geq 0 \qquad (L)$$

The key observation is that if A deviates and choose $e = 0$, (almost) surely $Y = 0$. If θ is unknown, A must choose an arbitrary t. Now, the probability of picking $t = \theta$ or t^* while randomly choosing a t is 0 (since t is in $[0, 1]$). Hence, (IC) simply requires $s + b - c \geq s$. The formulation of (IR) is routine.

(b) Solve for the optimal contract and compute P's equilibrium payoff. [*Hint:* At the optimum, $s = 0$.]

Solution.

The solution is $b = c$ and $s = 0$; P's payoff is $y - c$.

(c) Now suppose P reveals to A which task is t^*. Write down P's optimal contracting problem that implements $e = 1$. Clearly **explain** the (IC) and (IR) constraints and solve for the optimal contract. What is P's equilibrium payoff? (You must explain your derivation of the (IC) to be consider for any credit in this question).

Solution.

If t^* is known, P's problem is:

$$\begin{aligned} \max_{s,b} \quad & y - s - b && s.t. \\ b - c &\geq pb && (IC) \\ s + b - c &\geq 0 && (IR) \\ s \geq 0, b &\geq 0 && (L) \end{aligned}$$

The key observation is that if A deviates and choose $e = 0$, he would now choose t^* and obtain $Y = y$ with probability p. Hence, (IC) requires $s + b - c \geq s + pb$. The formulation of (IR) is routine. The solution is $b = \frac{c}{1-p}$ and $s = 0$; P's payoff is $y - \frac{c}{1-p}$.

(d) It is often argued that more transparency in organization is better for the business. Using your findings in (ii) and (iii), show that, contrary to this argument, letting A know about t^* hurts P and if p is large, no effort could be induced in equilibrium. Also, provide a brief intuition behind your finding.

Solution.

Note that P's payoff decreases in p and if $y - \frac{c}{1-p} < 0$ or $p > 1 - \frac{c}{y}$, P's payoff from implementing $e = 1$ is negative. The intuition is as follows: when t^* is known to A, it is easier for him to deviate as he knows which task to pick if θ is unknown. Hence (IC) becomes harder to satisfy and b must increase. But due to liquidity constraint $(s \geq 0)$, a higher b means more rent for A and less rent for P. At the extreme, satisfying (IC) becomes so costly that it does not cover the benefit y.

3. (MSU Prelim SS 2017 Q4) Consider a three-action, two-outcome principal-agent problem. The outcomes are denoted by $\pi_H = 10$ and $\pi_L = 0$, which represent gross profits to the principal. The three effort levels are denoted by as $a_H = \frac{20}{12}$, $a_M = \frac{19}{12}$ and $a_L = \frac{16}{12}$. The agent's utility function is of the form $U(w, a) = \sqrt{w} - a$, where a denotes effort, and w is the wage payment. Suppose that the the agent's reservation utility is 0. The effort level a_i results in outcome π_j with probability $f(\pi_j \mid a_i)$, where $f(\pi_H \mid a_H) = \frac{2}{3}$, $f(\pi_H \mid a_M) = \frac{1}{2}$, and $f(\pi_H \mid a_L) = \frac{1}{3}$.

   (a) What is the optimal contract when effort is observable?

   Solution.
   When the effort is observable, the principal can directly pay for the cost of the effort. So the principal's problem is

$$\underset{w=a^2}{argmax} \left\{ \frac{2}{3} \cdot 10 - \left(\frac{20}{12}\right)^2, \frac{1}{2} \cdot 10 - \left(\frac{19}{12}\right)^2, \frac{1}{3} \cdot 10 - \left(\frac{16}{12}\right)^2 \right\} = a_H^2$$

   So the optimal contract when effort is observable is to pay $\left(w = a_H^2, a_H = \frac{20}{12}\right)$

   (b) What is the optimal contract when effort is not observable? [Hint: Not all effort levels may be implementable.

   Solution.
   In the case where effort is not observable, the firm can only pay a wage w_H when π_H is realized and a wage w_L when π_L is realized. For the optimal contract, we need to solve this problem with backward induction. First step is to see if all of these effort levels are implementable. The expected utility of each effort level is:

$$\begin{aligned} a_H &: \frac{2}{3}\sqrt{w_H} + \frac{1}{3}\sqrt{w_L} - \frac{20}{12} \\ a_M &: \frac{1}{2}\sqrt{w_H} + \frac{1}{2}\sqrt{w_L} - \frac{19}{12} \\ a_L &: \frac{1}{3}\sqrt{w_H} + \frac{2}{3}\sqrt{w_L} - \frac{16}{12} \end{aligned}$$

If a_H is implementable, then IC_H must be satisfied, meaning:

$$\frac{2}{3}\sqrt{w_H} + \frac{1}{3}\sqrt{w_L} - \frac{20}{12} \geq \frac{1}{2}\sqrt{w_H} + \frac{1}{2}\sqrt{w_L} - \frac{19}{12} \iff \frac{1}{6}\sqrt{w_H} - \frac{1}{6}\sqrt{w_L} \geq \frac{1}{12}$$
$$\frac{2}{3}\sqrt{w_H} + \frac{1}{3}\sqrt{w_L} - \frac{20}{12} \geq \frac{1}{3}\sqrt{w_H} + \frac{2}{3}\sqrt{w_L} - \frac{16}{12} \iff \frac{1}{3}\sqrt{w_H} - \frac{1}{3}\sqrt{w_L} \geq \frac{4}{12}$$

If a_M is implementable, then IC_M must be satisfied, meaning:

$$\frac{2}{3}\sqrt{w_H} + \frac{1}{3}\sqrt{w_L} - \frac{20}{12} \leq \frac{1}{2}\sqrt{w_H} + \frac{1}{2}\sqrt{w_L} - \frac{19}{12} \iff \frac{1}{6}\sqrt{w_H} - \frac{1}{6}\sqrt{w_L} \leq \frac{1}{12}$$
$$\frac{1}{2}\sqrt{w_H} + \frac{1}{2}\sqrt{w_L} - \frac{19}{12} \geq \frac{1}{3}\sqrt{w_H} + \frac{2}{3}\sqrt{w_L} - \frac{16}{12} \iff \frac{1}{6}\sqrt{w_H} - \frac{1}{6}\sqrt{w_L} \geq \frac{3}{12}$$

But these two conditions are inconsistent with each other, so a_M must not be implementable. As such, we need only to consider implementing a_H or a_L
If the principal wants to implement a_L, they simply pay $w = a_L^2 = \left(\frac{16}{12}\right)^2$, the expected profit is:

$$\frac{10}{3} - \frac{16}{9} = \frac{14}{9}$$

If the principal wants to implement a_H, then they need to make sure IR_H and IC_H hold:

$$\frac{1}{6}\sqrt{w_H} - \frac{1}{6}\sqrt{w_L} \geq \frac{1}{12} \iff \sqrt{w_H} - \sqrt{w_L} \geq \frac{1}{2}$$
$$\frac{1}{3}\sqrt{w_H} - \frac{1}{3}\sqrt{w_L} \geq \frac{4}{12} \iff \sqrt{w_H} - \sqrt{w_L} \geq 1$$
$$\frac{2}{3}\sqrt{w_H} + \frac{1}{3}\sqrt{w_L} \geq \frac{20}{12} \iff 2\sqrt{w_H} + \sqrt{w_L} \geq 5$$

The optimal contract would have the second and third condition bind. Solving for this we get $\sqrt{w_H} = 2, \sqrt{w_L} = 1$. So to implement a_H, the principal needs to pay $w_H = 4$, $w_L = 1$. The expected profit is thus:

$$\frac{2}{3}(10-4) + \frac{1}{3}(-1) = 4 - \frac{1}{3} = \frac{11}{3} = \frac{33}{9} > \frac{14}{9}$$

As such, the optimal contract is $w_H = 4$, $w_L = 1$ and the principal wants to implement a_H

4. (MSU SS 2013 Part II Q4) Consider a two person partnership problem. Output is generated by the simultaneously chosen efforts of two agents. Specifically, if agent 1 chooses $e_1 \geq 0$ and agent 2 chooses $e_2 \geq 0$, their joint output is a deterministic function $F(e_1, e_2)$ of their efforts. Each agent i bears a cost of her own effort of $c(e_i) = \frac{1}{2}e_i^2$. The output is publicly observable.

   (a) Suppose $F(e_1, e_2) = e_1 + e_2$ and that the effort is observable. Find the first-best effort choices (e_1^*, e_2^*), i.e., those that maximize the output net of the agents' costs.

   Solution.

   In first best, effort is observable. As stated by the question, we need to find effort level that maximizes the output net of costs. So the following F.O.C.s must be satisfied:

$$F_{e_1} = e_1$$
$$F_{e_2} = e_2$$

   This means that $e_1^{FB} = e_2^{FB} = 1$.

(b) Suppose $F(e_1, e_2) = e_1 + e_2$ as before, but the effort is not observable. Suppose further that output must be shared in a budget balancing way, i.e., if the output turns out to be x and agent 1 receives $s(x)$, then agent 2 receives $x - s(x)$. The agents choose effort levels that form a Nash equilibrium of the game with payoffs

$$u_1(e_1, e_2) = s(F(e_1, e_2)) - c(e_1)$$
$$\text{and } u_2(e_1, e_2) = F(e_1, e_2) - s(F(e_1, e_2)) - c(e_2)$$

Assume that $s(\cdot)$ is differentiable. Show that the first best cannot be an outcome of any pure-strategy Nash equilibrium.

Solution.

Suppose otherwise that there exists some e_1^* and e_2^* that are PSNEs in this case. Then it must be that u_1 and u_2 are maximized, meaning the the following F.O.C.s must be satisfied:

$$\frac{\partial}{\partial e_1} u_1(e_1, e_2) = s'(F^*)F_{e_1}(e_1^*, e_2^*) - e_1^* = s'(F^*) - e_1^* = 0$$
$$\frac{\partial}{\partial e_2} u_2(e_1, e_2) = F_{e_2}(e_1^*, e_2^*) - s'(F^*)F_{e_2}(e_1^*, e_2^*) - e_2^* = [1 - s'(F^*)] - e_2^* = 0$$

These F.O.C.s yield:

$$e_1^* = s'(F^*) \text{ and } e_2^* = 1 - s'(F^*)$$

The first-best outcome is $e_1^{FB} = e_2^{FB} = 1$, which will violate this condition. As such, first-best outcome is not achievable in this case.

(c) Now suppose that $F(e_1, e_2) = \min\{e_1, e_2\}$, and the cost functions are unchanged. Find the first-best allocation, and show that it can be achieved in a Nash equilibrium with a sharing rule according to which each agent receives half of the output, i.e., $s(x) = \frac{1}{2}x$.

Solution.

The first-best problem maximizes the total output less the total costs:

$$\max_{e_1, e_2 \geq 0} \quad \min\{e_1, e_2\} - \frac{1}{2}e_1^2 - \frac{1}{2}e_2^2$$

Assume that we have $e_i > e_j$ in the optimum. Then the total payoff is $e_j - \frac{1}{2}e_i^2 - \frac{1}{2}e_j^2$. But we can increase the total payoff by reducing e_i to the level of e_j: $e_j - e_j^2$. Thus $e_1 = e_2 = e$ in the optimum, and the problem becomes

$$\max_{e \geq 0} \quad e - e^2$$

which is maximized at $e = \frac{1}{2}$.

Let us show that these first best efforts form a Nash equilibrium under the equal linear sharing rule. Consider the problem of the agent i is to maximize own payoff taking the effort of agent j ($\hat{e}_j = \frac{1}{2}$) as given:

$$\max_{e_i \geq 0} \quad \frac{1}{2}\min\left\{e_i, \frac{1}{2}\right\} - \frac{1}{2}e_i^2$$

The objective is increasing on $[0, \frac{1}{2}]$, and decreasing on $\left(\frac{1}{2}, \infty\right)$. Hence the maximum is achieved at $e_i = \frac{1}{2}$, and $(e_1, e_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$ is a Nash equilibrium.