[SP26] ECN 812B Recitation 3

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1 Concepts this Week

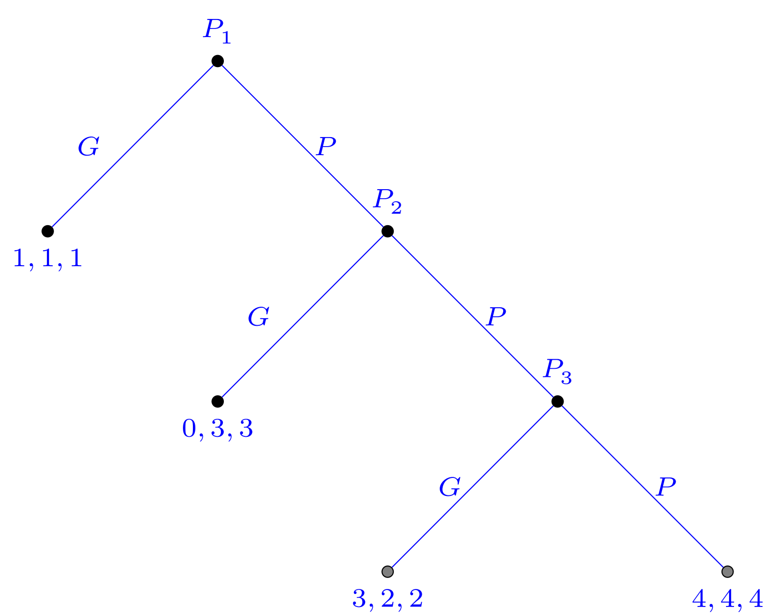
- Dynamic/Sequential Games
  - Subgame: A Subgame begins at a singleton information set and contains all and only the successor nodes of this node.
  - Backward Induction: Solve for NEs in all the lowest level subgames, then reduce those subgames to deterministic nodes. Iterate until the reduced full game is the only subgame.
  - Sequential Rationality: A player's strategy is sequentially rational if it plays a best response to other players' strategies at all information sets in the game (We will revisit this later for some refinement).
- Subgame Perfect Nash Equilibrium: A *strategy profile* is an SPNE if it is an NE in every subgame.
  - In finite strategic games, an SPNE always exists
- Finite Horizon Nash Bargaining: An application of backward induction/SPNE.
- Infinite Horizon Nash Bargaining: Backward induction doesn't apply $\Rightarrow$ specify expected payoff for some period t and backward induct so that you can draw equality between discounted payoff and equilibrium payoff.

2 Learning by Doing

1. (MSU Midterm 2023) Consider a sequential move centipede game with three players. Each player in turn may either "grab" or "pass". The game ends when a player grabs, or when the final mover passes. If they all pass, they all get 4. If the first mover grabs they all get 1. If the first mover passes and the second mover grabs, the payoffs are $(0, 3, 3)$, where the i^{th} number denotes the payoff of player i. If the first and second mover both passed, but the third mover grabs the payoffs are $(3, 2, 2)$.

   (a) Draw the corresponding game tree for this game.

   Solution.



   (b) Find the unique subgame perfect Nash equilibrium of this game.

   Solution.
   Using backwards induction, in the last proper subgame, P_3's BR is Pass.
   Reducing the game, P_2's BR is Pass.
   Reducing the game, P_1's BR is Pass.
   The unique SPNE is (P, P, P).

(c) Find all Nash equilibria. Show that there is no Nash equilibrium, mixed or otherwise, in which there is a positive probability that the first mover passes and the second mover grabs.

Solution.

The payoff table is the following:

		P_2		P_2	
		G	P	G	P
P_1	G	$\underline{1}, \underline{1}, \underline{1}$	$1, \underline{1}, \underline{1}$	$\underline{1}, \underline{1}, \underline{1}$	$1, \underline{1}, \underline{1}$
	P	$0, \underline{3}, \underline{3}$	$\underline{3}, 2, 2$	$0, 3, \underline{3}$	$\underline{4}, \underline{4}, \underline{4}$
	P_3	G		P	

So the set of PSNE is $\{(G, G, G), (G, G, P), (P, P, P)\}$.

Given that P_3's best response in stage 3 is P, G is strictly dominated (in the reduced game) by P for P_2. As such, we know that any strategy with P_2 playing G with strictly positive probability when P_1 passes is strictly dominated, and hence not an NE.

2. (Modified from MSU Prelim SS 2023) Consider a Cournot model with three firms that have no production cost (i.e. the marginal cost for each firm is zero). The inverse demand function in the market is $P(Q) = 12 - Q$.

   (a) What is the Nash equilibrium of the (simultaneous move) game? What are the profits for each firm equilibrium?

   Solution.
   This is the quantity competition model with no marginal cost of production and demand $P(Q) = 12 - Q$. Firm i's problem is

$$\max_{q_i} (12 - q_i - q_j - q_k)q_i$$

   The F.O.C. is

$$[q_i] : 12 - 2q_i^* - q_j - q_k = 0 \Rightarrow BR_i(q_j, q_k) = 6 - \frac{q_j + q_k}{2}$$

   By symmetry, firms j and k have the same best responses, so in NE, we have

$$q_i^* = q_j^* = q_k^* = q^* = 6 - q^* \Rightarrow q^* = 3 \Rightarrow \pi^* = 9$$

   The NE is $\{(q_1, q_2, q_3) = (3, 3, 3)\}$ and their profits are $\{(\pi_1, \pi_2, \pi_3) = (9, 9, 9)\}$

(b) Suppose there are three potential entrants to the market and that there is a entry cost of $K = 33$, which needs to be paid to enter the market. Suppose that all three firms make entry decisions simultaneously. Find the symmetric mixed strategy Nash equilibrium for entry decisions (that is, what is the probability of entry by each firm)?

Solution.

Assuming that competition is standard Cournot (as opposed to collusion) after entry decisions are made. We already know the profit of firms if all 3 firms enter. Now, we shall check if 2 firms enter, and if 1 firm enters.

For 2 firms, we know the best response of firm i is

$$BR_i(q_j) = 6 - \frac{q_j}{2}$$

So in equilibrium, $q_i^D = q_j^D = q^D = 4$ and $\pi^D = 16$. If 1 firm enters, we know the monopoly quantity is $q^M = 6$ add the profit is $\pi^M = 36$.

In the symmetric MSNE, it must be that all firms are mixing in a way such that other firms are indifferent between entry and no entry. Let the all firms enter with probability p in the symmetric MSNE, then it must be that:

$$(1-p)^2 \cdot 36 + 2p(1-p) \cdot 16 + p^2 \cdot 9 - 33 = 0 \iff p = \frac{1}{13}$$

So the symmetric MSNE is

$$\sigma_1 = \sigma_2 = \sigma_3 = \frac{1}{13}\text{Enter} + \frac{12}{13}\text{No Enter}$$

(c) Now suppose that the entry decision is made sequentially (firm 1 makes its entry decision first, then firm 2, and then firm 3). The entry decision made by each firm is observable. What would be the equilibrium entry configuration (i.e., which firms, if any, enter the market?)

Solution.

By backwards induction, firm 3 will only enter if

$$\pi_3 = (12 - Q)q_3 \geq 33$$

But by the previous question, we know that if all 3 firms enter, the equilibrium profit is only 9. If 2 firms enter, the equilibrium profit is only 16. As such, in an SPNE, firm 3 will enter if and only if firms 1 and 2 did not.

Similarly, in an SPNE firm 2 will enter if and only if firm 1 did not.

Knowing this, firm 1 will enter if $\pi^M - 33 \geq 0$, so firm 1 will enter the market.

By backwards induction, the unique SPNE is one where only firm 1 enters.

3. (MSU Prelim FS 2015 Part 1 Q4) Consider the following version of the "ultimatum game": Two players decide on how to split a dollar. Player 1 moves first and proposes a split $(s, 1-s)$. Player 2 accepts or rejects. If player 2 accepts, then player 1 gets s and 2 gets $1-s$. If player 2 rejects, then both players get 0. But as opposed to the typical utility function assumed in this game, suppose that the players care about "fairness" and has the following utility function. For any payoff profile (x_i, x_j), player i's utility is:

$$u_i(x_i, x_j) = x_i - \alpha_i \max\{x_j - x_i, 0\} - \beta_i \max\{x_i - x_j, 0\}$$

where $0 \le \beta_i \le \alpha_i \le 1$. (That is, players don't like inequality (as both α_i and β_i are positive), and they dislike being behind more than being ahead.) Under these preferences, find the SPNE of the ultimatum game described above. [Hint: Player 1's offer would depend on whether $\beta_1 > \frac{1}{2}$ or not.]

Solution.

By backward induction. If P_1 offers $(s, 1-s)$, P_2's utility payoff if she accepts is:

$$u_2(1-s, s) = (1-s) - \alpha_2 \max\{2s-1, 0\} - \beta_2 \max\{1-2s, 0\}$$

As such, P_2's best response is:

$$BR_2(1-s, s) = \begin{cases} \text{Accept} & \text{if } s > \frac{1}{2} \wedge 1-s-\alpha_2(2s-1) \ge 0 \\ \text{Accept} & \text{if } s = \frac{1}{2} \\ \text{Accept} & \text{if } s < \frac{1}{2} \wedge 1-s-\beta_2(1-2s) \ge 0 \\ \text{Decline} & \text{otherwise} \end{cases}$$

P_1's utility is thus

$$u_1(s, 1-s) = \begin{cases} s - \beta_1(2s-1) & \text{if } s > \frac{1}{2} \wedge 1-s-\alpha_2(2s-1) \ge 0 \\ \frac{1}{2} & \text{if } s = \frac{1}{2} \\ s - \alpha_1(1-2s) & \text{if } s < \frac{1}{2} \wedge 1-s-\beta_2(1-2s) \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Notice that $s < \frac{1}{2}$ is strictly dominated by $s = \frac{1}{2}$, so we need only to check $s \ge \frac{1}{2}$.
Now, $\forall s \in \left[\frac{1}{2}, \frac{1+\alpha_2}{1+2\alpha_2}\right]$, $\frac{d}{ds}u_1(s, 1-s) = 1 - 2\beta_1 > 0 \iff \beta_1 < \frac{1}{2}$.
SPNE is $s = \frac{1}{2}$ if $\beta_1 > \frac{1}{2}$ and $s = \frac{1+\alpha_2}{1+2\alpha_2}$ if $\beta_1 \le \frac{1}{2}$, and P_2 always accepts.

4. (BU Prelim SS 2019 Q2) Consider the following three-person bargaining game in which players 1, 2, and 3 bargain over how to divide a surplus of size 1. Player 1 makes the first offer, denoted $x^1 = (x_1^1, x_2^1, x_3^1) \in \mathbb{R}_+^3$ with $x_1^1 + x_2^1 + x_3^1 = 1$. Next, player 2 either accepts or rejects. If player 2 accepts, then it is player 3's turn to either accept or reject. If player 3 also accepts, the game is over and the outcome is x^1. If either player 2 or 3 rejects, the game goes to period 2 and it becomes player 2's turn to make an offer with player 3 responding first and player 1 responding second. If player 3 and 1 accept player 2's offer, then the outcome is the offer x^2 made by player 2. If not, the game goes to period 3 with player 3 making the offer, player 1 responding first, and player 2 responding second. The game continues this way until the players reach an agreement. If the game ends with agreement on an offer of $x = (x_1, x_2, x_3)$ where this is the t^{th} offer made, then player $i's$ payoff is $\delta^t x_i$ where $\delta \in (0, 1)$ is the common discount factor.

   (a) Show that the game has a SPNE in which at each period t, if players still have not reached an agreement, the proposer would offer $\frac{\delta}{1+\delta+\delta^2}$ to the first responder, $\frac{\delta^2}{1+\delta+\delta^2}$ to the second responser, and keeps $\frac{1}{1+\delta+\delta^2}$ for themselves.

   Solution.
   Consider the strategy profile given. First, notice that the first responder is also the one to make the offer in the next period if the bargaining falls through in this period. As such, the decision the first responder makes is

$$\max\left\{\frac{\delta}{1+\delta+\delta^2}\ ,\ \delta\cdot\frac{1}{1+\delta+\delta^2}\right\}$$

   Similarly, the third responder will make the same decision next period, and so the problem they face this period is:

$$\max\left\{\frac{\delta^2}{1+\delta+\delta^2}\ ,\ \delta\cdot\delta\cdot\frac{1}{1+\delta+\delta^2}\right\}$$

   Since neither player has an incentive to unilaterally deviate, the proposed strategy profile is indeed an SPNE.

(b) Now consider a variation of this game. As before, players alternate in their right to make offers, with player 1 making offers first, followed by player 2, and then player 3. The difference is that, at the start of each period, the proposer has the opportunity to walk out of the bargaining. If the proposer does NOT walk, they make an offer like before. If they walk, the game ends and the proposer gets an outside option payoff of $y \in \left(\frac{1}{1+\delta}, 1\right)$, which the other two players get 0. Find a SPNE of this game.

Solution.

Since the second responder would become the first if the game goes to the next period, we can recursively solve for them by looking at the first responder. Let player 1 make the offer at some period t subgame. The first responder's problem is:

$$\max\{x_2^1 \ , \ \delta y \ , \ \delta x_2^2\}$$

The second responder's problem is then:

$$\max\{x_3^1 \ , \ \delta^2 y \ , \ \delta^2 x_3^2\}$$

If player 1 wants the offer to be accepted by both, it must be that

$$x_2^1 \geq \max\{\delta y \, , \, \delta x_2^2\} \ , \ x_3^1 \geq \max\{\delta^2 y \, , \, \delta^2 x_3^2\}$$

This means that:

$$x_1^1 = 1 - x_2^1 - x_3^1 \leq 1 - \delta y - \delta^2 y$$

$$\text{or}$$

$$x_1^1 = 1 - x_2^1 - x_3^1 \leq 1 - \delta x_2^2 - \delta^2 x_3^2 < 1 - \delta y - \delta^2 y$$

From the question we know that $y > \frac{1}{1+\delta}$, meaning $y > 1 - \delta y$. So we have

$$y > 1 - \delta y > 1 - \delta y - \delta^2 y \geq x_1^1$$

As such, player 1's best response is to walk out in the beginning. Player 2's best response is to reject and walk out when they propose, and player 3's best response is to reject twice and walk out when they propose. This strategy profile constitutes an SPNE.

3 Go the Extra Mile

1. (MSU 2020 Midterm 1 Q1) Consider the following two-player, three-action game.

P_1 \ P_2	L	C	R
U	2, 1	1, 0	1, 0
M	1, 0	1, 0	1, 0
D	1, 0	1, 0	1, 0
B	-1, 0	0, 1	0, 1

   (a) Identify the set of strictly dominated strategies for each player.

   (b) Identify the set of rationalizable strategies for each player.

   (c) Identify all pure strategy Nash Equilibria.

   (d) Identify all mixed strategy Nash Equilibria

2. (MSU Prelim FS 2012 Part II Q1) Two workers A and B work together on a product development project. Each of them must decide how much effort to put into the project. The cost of effort $e > 0$ is equal to $c(e) = e^2$ for each one of them. The quality of the project, and their joint profits depend on their joint efforts. If A chooses e_A and B chooses e_B, then their expected joint profits are equal to

$$\pi(e_A, e_B) = e_A^{1/2} + e_B^{1/2}.$$

   After they choose their effort, they enter two rounds of alternating offer bargaining about the profits. A makes the first offer, and if the offer is rejected, B makes the offer. If A's offer is rejected, the joint profits fall to $\frac{1}{2}\pi(e_A, e_B)$. If B's offer is rejected, the joint profits fall to 0 and the game ends. (For example, think about a situation where any delay in implementing the project increases the chance that a competitor comes up with a better product and takes away their demand). What are the subgame perfect equilibrium levels of effort?