[SP26] ECN 812B Recitation 5

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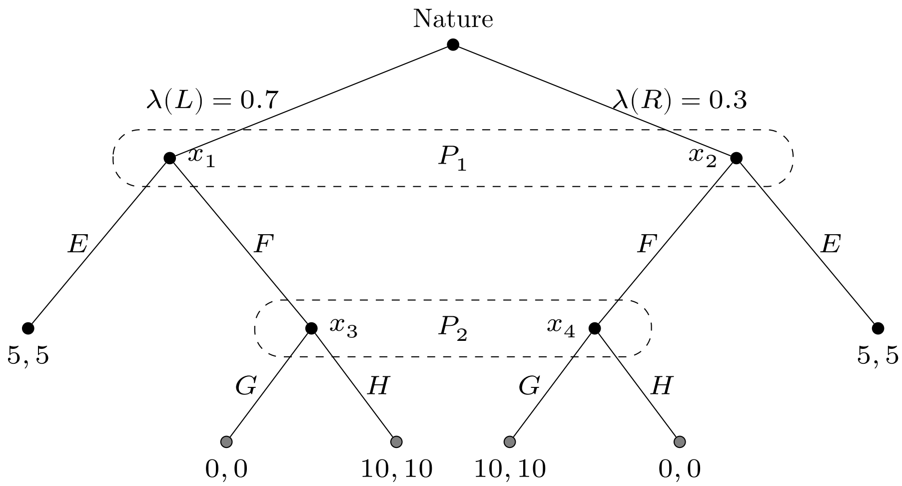
1 Concepts this Week

- A **system of beliefs** μ is a specification of probability distribution over nodes within an information set.
- A strategy profile σ is **sequentially rational** given the belief system μ if σ is the SPNE when μ is used to calculate the expected payoffs.
  - Note that whether a strategy profile is sequentially rational or not have nothing to do with whether the system of belief μ makes sense (other than summing to 1 within an information set)
  - If the system of belief makes sense, we say that it (the system of beliefs) is **consistent**
  - Next we will cover equilibrium concepts where the system of beliefs is consistent only in parts of the game (wPBE) and in all of the game (PBE).
- The tuple (σ, μ) where σ is a strategy profile and μ is a system of beliefs is a **weak Perfect Bayesian Equilibrium** if
  - The strategy profile is sequentially rational given the system of beliefs
  - The system of beliefs is consistent (via Bayes' rule) given the strategy profile *on the path of equilibrium*
- **Perfect Bayesian Equilibrium**: A tuple (σ, μ) is a PBE if σ is sequentially rational given μ and μ is consistent in every subgame
- **Sequential Equilibrium**: A tuple (σ, μ) is an SE if σ is sequentially rational given μ and μ is consistent in every subgame and there exists a sequence of tuples $(\sigma^k, \mu^k) \to (\sigma, \mu)$
  - For sequential equilibrium, the sequence σ^k need NOT be sequentially rational, but σ does. If σ^k is sequentially rational for all k and μ^k is consistent for all k, then (σ, μ) can be further refined as an *Extensive-Form THPNE* (out of the scope of this course).

- Logic chain for solving a PBE/SE problem:



See that P_2's only consistent belief is $\mu(x_3) = 0.7$ if F is played with non-zero probability. Next, see that P_2's best response is to play G with probability 1 if $\mu(x_4) \geq \frac{1}{2}$ and play H with probability 1 if $\mu(x_3) \geq \frac{1}{2}$.

If $\mu(x_3) > \frac{1}{2}$, P_1's expected payoff for playing F is $0.7 \cdot 10 + 0.3 \cdot 0 = 7 > 5$. If $\mu(x_3) < \frac{1}{2}$, then P_1's expected payoff for playing F is 3. So our (pure strategy) PBEs (and hence SE candidates) are

$$((F, H), \mu(x_3) = 0.7) \text{ and } ((E, G), \mu(x_3) < \frac{1}{2})$$

Take the tuple $((E, G), P(x_3) = 0.3)$, we know this is a PBE. But it is not an SE, because for any sequence of behavior/mixed strategies such that F is played with strictly positive probability ε_k, the consistent belief would be

$$\mu^k(x_3) = \frac{0.7\varepsilon_k}{0.7\varepsilon_k + 0.3\varepsilon_k} = 0.7 \not\to 0.3$$

Since the sequence of beliefs does not converge to 1 as the sequence of strategies converge to (E, G), the specified tuple is not an SE.
With the same logic, the tuple $((F, H), \mu(x_3) = 0.7)$ is an SE of this game.

2 Learning by Doing

1. (MSU Prelim SS 2019 Q3) Consider an expert (E) who advises a decision maker (DM) about the underlying state of the world $\theta \in \{0,1\}$. DM does not observe θ but E privately observes an informative but noisy signal $s \in \{0,1\}$. The precision of the signal is given by $Pr(s = k \mid \theta = k) = q$, for $k = 0, 1$, and $q > \frac{1}{2}$. Both players hold a common prior over the state given by $Pr(\theta = 1) = \mu \in (0,1)$. The game unfolds in the following four stages: First, nature chooses θ according to the prior. Next, E observes s and reports $m \in \{0,1\}$ to DM. Then, DM takes an action $x \in \{0,1\}$. Finally, θ is observed and payoffs are realized. The payoff of DM is given by:

$$U(x,\theta) = \left(\theta - \frac{1}{2}\right) x$$

and the payoff of E is given by:

$$V(m,\theta) = \mathbb{1}\{m = \theta\}$$

That is, DM wants to tailor her action to the underlying state, but E only cares about his reputation on "getting the state right."
Construct a PBE of this game where E always reports his signal truthfully. Do you need any restrictions on the parameters μ and q for such a PBE to exist? If so, clearly state this condition, and provide a brief economic intuition for it.

Solution.
Let's first look at the conditions on (μ, q) for E reporting truthfully being sequentially rational.
When $s = 0$, E will truthfully report if:

$$\begin{aligned}
&P(\theta = 0 \mid s = 0) \geq P(\theta = 1 \mid s = 0) \\
\iff &\frac{(1-\mu)q}{\mu(1-q) + (1-\mu)q} \geq \frac{\mu(1-q)}{\mu(1-q) + (1-\mu)q} \\
\iff &q \geq \mu
\end{aligned}$$

When $s = 1$, E will truthfully report if:

$$\begin{aligned} & P(\theta = 1 \mid s = 1) > P(\theta = 1 \mid s = 1) \\ \iff & \frac{\mu q}{\mu q + (1-\mu)(1-q)} \geq \frac{(1-\mu)(1-q)}{\mu q + (1-\mu)(1-q)} \\ \iff & q \geq 1 - \mu \end{aligned}$$

As such, always truthfully reporting is sequentially rational (for E) if and only if

$$q \geq \max\{\mu, 1-\mu\}$$

Suppose that this condition is satisfied, then DM should choose $x = 0$ when $E[\theta \mid m] \leq \frac{1}{2}$ and $x = 1$ when $E[\theta \mid m] \geq \frac{1}{2}$ in a sequentially rational strategy.
If DM observes $m = 0$, DM will choose $x = 0$ if and only if

$$E[\theta \mid m = 0] = P(\theta = 1 \mid m = 0) = \frac{\mu(1-q)}{\mu(1-q) + (1-\mu)q} \leq \frac{1}{2} \iff q \geq \mu$$

Similarly, if DM observes $m = 1$, DM will choose $x = 1$ if and only if

$$E[\theta \mid m = 1] = P(\theta = 1 \mid m = 1) = \frac{\mu q}{\mu q + (1-\mu)(1-q)} \geq \frac{1}{2} \iff q \geq 1 - \mu$$

So DM will play exactly what the message is if and only if

$$q \geq \max\{\mu, 1-\mu\}$$

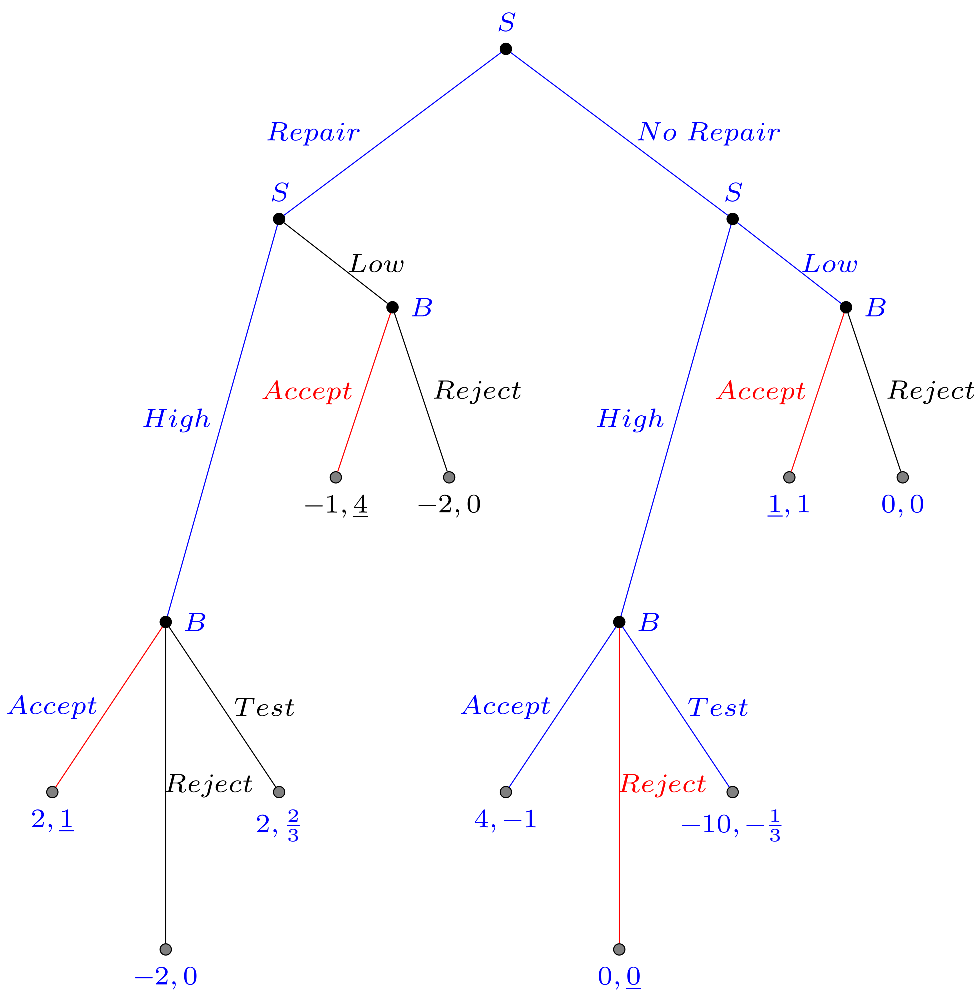
As such, a PBE where E always truthfully reports (and DM subsequently acting according to the message) requires that the signal is sufficiently precise. The PBE is:

$$((m(s) = s,\, x(m) = m), P(\theta \mid m) = P(\theta \mid s))$$

2. (Modified from UChicago EC302 PS8) A seller (S) of a used car can either repair it prior to resale or not. Either way, he can demand either a high or a low price for the car. The buyer (B) can either accept the price, reject the price or test the car. But it is never worth testing if the price is low and so this option is simply not considered. This situation is depicted in the following extensive form game (Payoff structure is U_S, U_B).

   (a) First, suppose that the game is actually of perfect information as depicted in the game tree below. Find all the SPNEs.

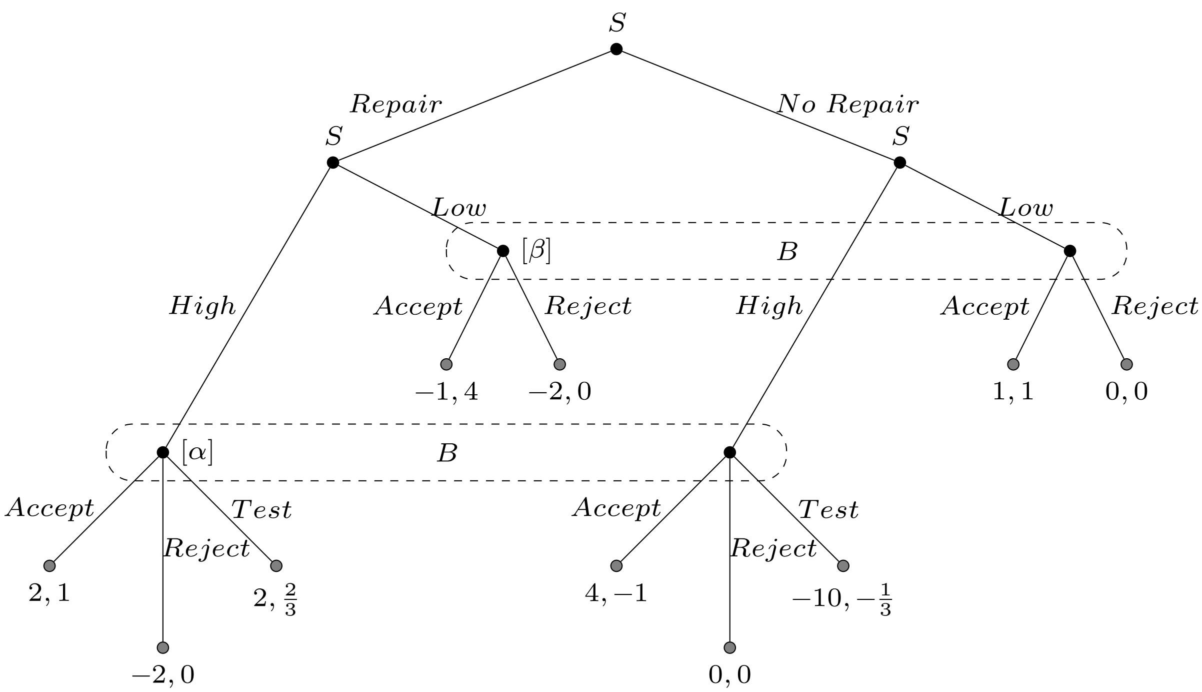
Solution.



Using backward induction, the unique SPNE of this sequential game of perfect information is

$$(Repair\text{—}High_R\text{—}Low_{NR},\ Accept_{R-H}\text{—}Accept_{R-L}\text{—}Reject_{NR-H}\text{—}Accept_{NR-L})$$

(b) Suppose that the buyer is naive and believes that the seller did the repairs with probability $\alpha = 1$ if the price is high and probability $\beta = \frac{1}{2}$ if price is low. Find the unique strategy profile that is sequentially rational.



Solution.

First, given the structure of the game has now changed, we should note that the seller's strategy space has not changed, but the buyer's strategy space has. The buyer now has 6 pure strategies instead of the 36 in part (a). Given the system of belief provided in the question, the buyer's BR when facing high price is to accept. When facing low price, their expected utility if accept is $\frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 1 = \frac{5}{2}$ and their expected utility if reject is 0. As such, the buyer should always accept, given this system of beliefs.

Since the buyer will always accept, the seller's BR is then to **not repair** and then to **offer high price**. As such, $(NR - H_R - H_{NR},\ Accept_H - Accept_L)$ is the unique sequentially rational strategy profile given this system of beliefs.

(c) Is the strategy profile you found in part (b) part of a Weak Perfect Bayesian Equilibrium? Why or why not?

Solution.
No, it is not. If B always accepting when price is high is part of a wPBE, it must be sequentially rational, meaning the belief α must be that

$$\alpha \cdot 1 + (1-\alpha) \cdot (-1) \geq \max\left\{\alpha \cdot \frac{2}{3} + (1-\alpha) \cdot \left(-\frac{1}{3}\right), 0\right\} \;\Rightarrow\; \alpha \geq \frac{2}{3}$$

So for this profile to be part of a wPBE, the belief system must involve S repairing at least $\frac{2}{3}$ of the time conditional on the price being high. But this profile specifies that the seller will do so with 0 probability, so any $\alpha \geq \frac{2}{3}$ is not consistent. As such, $(NR - H_R - H_{NR},\ Accept_H - Accept_L)$ cannot be part of a wPBE.

(d) Now suppose that the buyer believes that the seller did the repairs with probability $\alpha \in [0, 1]$ if the price is high and probability $\beta = \frac{1}{2}$ if the price is low. Find a weak Perfect Bayesian Equilibrium where B accepts high price with a strictly positive probability.

Solution.
As we derived in part (c), if $\alpha = \frac{2}{3}$, then B could be willing to mix their strategy when offered a high price. Suppose that in such mixed strategy, B plays $Accept_H$ with probability p and $Test_H$ with probability $1-p$ (We can ignore the strategy $Reject_H$ as when $\alpha = \frac{2}{3}$, it is strictly dominated by $Accept_H$ and $Test_H$). Then, for S to mix $R - H$ with strictly positive probability, it must be that

$$p \cdot 2 + (1-p) \cdot 2 = p \cdot 4 + (1-p) \cdot (-10) \;\Rightarrow\; p = \frac{6}{7}$$

So if $p = \frac{6}{7}$, S can mix between $R - H$ and $NR - H$ (Notice that, due to the indifference condition, it does not matter to S how they mix). Lastly, notice that under B's strategy $\frac{6}{7}Accept_H + \frac{1}{7}Test_H - Accept_L$, the expected payoff of S picking $R - L$ or $NR - L$ are strictly less than their expected payoff if they played $\frac{2}{3}R - H + \frac{1}{3}NR - H$. As such, a wPBE (σ, μ) is:

$$\left(\left(\frac{2}{3}R - H_R - H_{NR} + \frac{1}{3}NR - H_R - H_{NR}, \frac{6}{7}Accept_H - Accept_L + \frac{1}{7}Test_H - Accept_L\right),\right.$$
$$\left.\left(\alpha = \frac{2}{3}, \beta = \frac{1}{2}\right)\right)$$

(e) Is the tuple (σ, μ) found in part (d) also a sequential equilibrium? If so, find a sequence of tuples $(\sigma^k, \mu^k) \to (\sigma, \mu)$. If not, find a belief system such that it is.

Solution.

Recall that to keep B mixing through the perturbation, it must be that $\alpha^k = \frac{2}{3}, \forall k$; otherwise, B's best response would be a pure strategy. Given that we want to check S's strategy $\frac{2}{3}R - H_R - H_{NR} + \frac{1}{3}NR - H_R - H_{NR}$, we can use $\lambda_S(R) = \frac{2}{3}$, $\lambda_S(H_R) = 1 - \varepsilon^k$, $\lambda_S(H_{NR}) = 1 - \delta^k$ as a sequence of S's behavior strategy and $\varepsilon^k \to 0$. The consistent beliefs would thus be

$$\alpha^k = \frac{\frac{2}{3}(1-\varepsilon^k)}{\frac{2}{3}(1-\varepsilon^k) + \frac{1}{3}(1-\delta^k)} \to \frac{2}{3}, \quad \beta^k = \frac{\frac{2}{3}\varepsilon^k}{\frac{2}{3}\varepsilon^k + \frac{1}{3}\delta^k}$$

It should be clear that what β^k converges to depends on the relationship between ε^k and δ^k. Take $\delta^k = 2\varepsilon^k$, then we have $\beta^k = \frac{1}{2}$. As such, the tuple

$$\left(\left(\frac{2}{3}R - H_R - H_{NR} + \frac{1}{3}NR - H_R - H_{NR}, \frac{6}{7}Accept_H - Accept_L + \frac{1}{7}Test_H - Accept_L\right), \left(\alpha = \frac{2}{3}, \beta = \frac{1}{2}\right)\right)$$

is indeed a sequential equilibrium.

Work that I accidentally did because I was thinking about extensive-form THPNE instead of SE.

Notice that $\forall k \; s.t. \; \lambda_S^k(H_R) \neq \lambda_S^k(H_{NR}) \;\Rightarrow\; \alpha^k \neq \frac{2}{3}$, so for the strategy profile to remain sequentially rational throughout the sequence, we must have $\beta^k = \alpha^k = \frac{2}{3}$. This means that the tuple in question is not an extensive-form trembling hand perfect equilibrium, even though it is a sequential equilibrium. Now we should see if the strategy profile in question is a EFTHPNE with the belief system $(\alpha, \beta) = (\frac{2}{3}, \frac{2}{3})$. Consider the strategy profile $\sigma^k = (\sigma_S^k, \sigma_B^k)$ where σ_S^k is the outcome equivalent of $\lambda_S(R) = \frac{2}{3}$, $\lambda_S(H_R) = 1 - \varepsilon^k$, $\lambda_S(H_{NR}) = 1 - \varepsilon^k$ and σ_B^k is the outcome equivalent of $\lambda_B(A_H) = \frac{6}{7} - \delta^k$, $\lambda_B(T_H) = \frac{1}{7}$, $\lambda_B(A_L) = 1 - \delta^k$. See that $\alpha^k = \beta^k = \frac{2}{3}$ is the sequence of consistent beliefs for this strategy profile, we should check that this profile is sequentially rational.

First, see that S's payoff of $R - H_R - \square$ and $NR - H_{NR} - \square$ are both $2 - 4\delta^k$, so the seller mixing between this two is still sequentially rational as long as $\delta_k \leq \frac{1}{4}, \forall k$ (based on payoff of $(NR - H_R - L_{NR}, \square - Accept_L)$). We also know that given this seller strategy and belief system, B's strategy as specified forms a best response for each k if $\varepsilon^k = \frac{1}{2k}$, $\delta_k = \frac{1}{5k}$. As such, the tuple $\left(\sigma, \mu(\alpha, \beta) = \left(\frac{2}{3}, \frac{2}{3}\right)\right)$ is a sequential equilibrium and further, an *Extensive-Form THPNE.*

(f) Find a sequential equilibrium in which the buyer rejects a high price with probability 1.

Solution.

If the B rejects a high price with probability 1, then α must be sufficiently low such that

$$u(Accept_H \mid \alpha) = 2\alpha - 1 \leq 0 = u(Reject_H \mid \alpha)$$
$$u(Accept_H \mid \alpha) = \alpha - \frac{1}{3} \leq 0 = u(Reject_H \mid \alpha)$$
$$\Rightarrow \ \alpha \leq \frac{1}{3}$$

If B is rejecting a high price with probability 1, then $R - \square - \square$ cannot be part of a sequentially rational strategy for S. Moreover, if B rejects high price with probability 1, the seller's best response is $NR - \square - L_{NR}$. Consider the behavior strategy $\lambda_S^k(R) = \varepsilon_k$, $\lambda_S^k(H_R) = \varepsilon_k$, $\lambda_S^k(H_{NR}) = \varepsilon_k$. The consistent beliefs based on this strategy is:

$$\alpha^k = \frac{\varepsilon^k \cdot \varepsilon^k}{\varepsilon^k \cdot \varepsilon^k + (1 - \varepsilon^k)\varepsilon^k} = \varepsilon^k \to 0$$
$$\beta^k = \frac{\varepsilon^k \cdot (1 - \varepsilon^k)}{\varepsilon^k \cdot (1 - \varepsilon^k) + (1 - \varepsilon^k) \cdot (1 - \varepsilon^k)} = \varepsilon^k \to 0$$

So the SE where B rejects a high price with probability 1 is

$$((Reject_H - Accept_L, NR - L_R - L_{NR}), (\alpha, \beta) = (0, 0))$$

3. (MSU Prelim FS 2015 Part II Q3) Consider the following buyer-seller game: The seller (S) has a product that she values at 0 and attempts to sell it to a buyer (B) who values it at v. However, v is B's private information. It is publicly known that $v \in \{1, 2\}$ where $Pr(v = 2) = \pi < \frac{1}{2}$. The game unfolds as follows: first, S posts a price $p \geq 0$. Knowing p, B decides whether to buy or not. If B buys, he gets a payoff of $v - p$ and S gets a payoff of p. If B declines to buy, both players get 0.

   (a) Find a PBE of this game. [Hint: What would be B's best response for any p?]

   Solution.
   B's best response is to buy if $v \geq p$ and not buy otherwise. If the seller posts $p = 2$, their expected payoff is $2\pi < 1$. If the seller posts $p = 1$, their expected payoff is 1. So the PBE is $((p = 1, Buy\mathbb{1}\{v \geq p\}), \mu(v = 1) = rt\pi)$

(b) Now suppose that the trade can take place over two periods $t = 0, 1$. The game is as follows: in period $t = 0$, S posts a price $p_0 \geq 0$. As before, if B buys, payoffs are $(p_0, v - p_0)$. But if he does not buy,the game goes to period $t = 1$. In $t = 1$, S posts another price $p_1 \geq 0$ and, again, B may buy at this price or refuse to trade. If B buys, period $t = 1$ payoffs are $(p_1, v - p_1)$. Both players discount future at rate $\delta \in (0, 1)$ (so, the present value of $t = 1$ payoff is $(\delta p_1, \delta(v - p_1))$). If B refuses to trade, the payoff is 0 for both players. Find the PBE of this game. Can trade take place in $t = 1$ in this environment?

Solution.

If B rejects in $t = 0$, S can form a belief $\alpha = P(v = 2 \mid B \text{ rejects in } t = 0)$. Notice that α need not be equal to the prior π since if $v = 2$ and $p = 2$, B has an incentive to not buy in $t = 1$ in order to mislead the seller into posting $p = 1$ in $t = 1$. If $v = 2$, let B_0 reject $p_0 > 1$ with probability $q \in [0, 1]$, then S's belief if B does not accept $p = 2$ would be

$$\alpha = \frac{\pi q}{\pi q + (1 - \pi)} \leq \pi < \frac{1}{2}$$

Then S's present discounted payoff if posting $p_1 = 2$ would be $\delta \cdot 2 \cdot \alpha$ and if posting $p_1 = 1$ would be $\delta < 1$. Since $\pi < \frac{1}{2}$, we know $\alpha < \frac{1}{2}$, $\forall q \in (0, 1)$, so $p_1 = 1$ is the best response for S in $t = 1$.

Now, if $v = 2$, and the buyer rejects in $t = 0$, then buyer should accept $p_1 = 1$, and they will get the payoff δ. This means that if S posts $p_0 = 2 - \delta$, the $v = 2$ buyer will not have an incentive to reject, making $q = 0$, and $\alpha = \pi$ part of a PBE.

Notice that S's payoff if posting $p_0 = 2 - \delta$ would be

$$\pi(2 - \delta) + \delta(1 - \pi) = \delta + \underbrace{2\pi(1 - \delta)}_{<1-\delta} < 1$$

So S"s sequentially rational strategy is $p_0 = 1, p_1 = 1$.

The PBE of this game this thus described by:

- Seller strategy: $p_0 = 1, p_1 = 1$
- Seller belief: $\alpha \leq \pi$

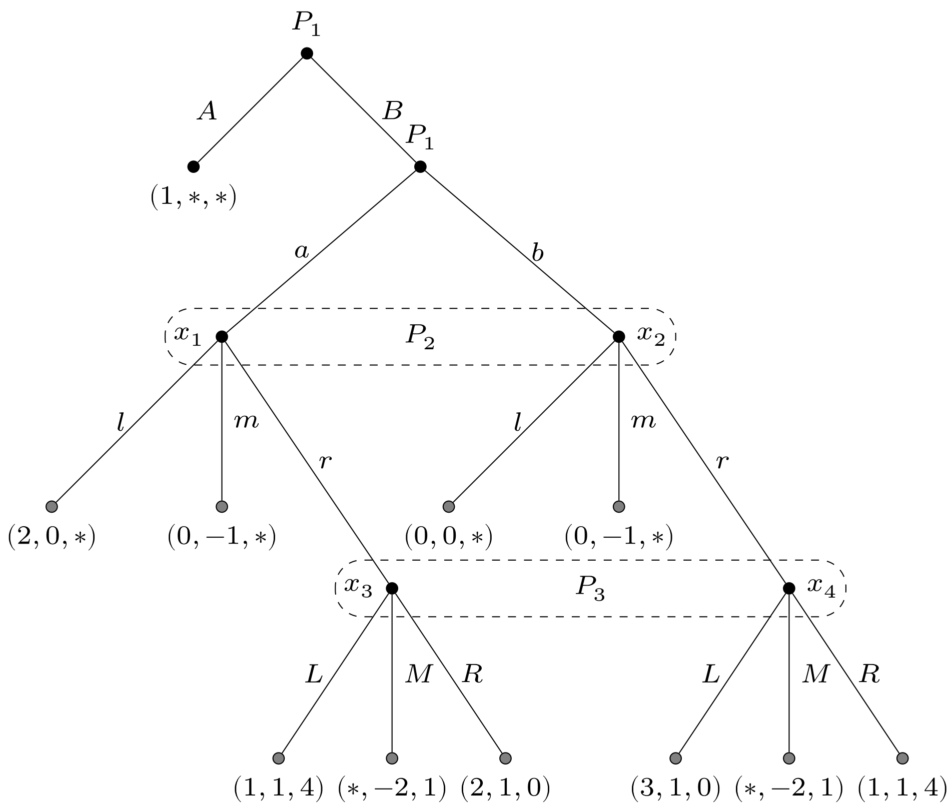
- Buyer strategy:

$$\begin{cases} Buy & p_0 = 1 \\ Buy & p_0 = 2 - \delta \ \wedge \ v = 2 \\ No\ Buy & Otherwise \end{cases}$$

Trade cannot take place in $t = 1$ in such environment. Since the seller cannot profitably give the $v = 2$ enough rent for them to reveal their type, the seller will simply choose $p_0 = 1$. From there, the buyer has no incentive to delay the trade (and every incentive to not delay, because of the discount rate δ).

3 Go the Extra Mile

1. (MSU Midterm 2019 Q2) Consider the following extensive form game.



An asterisk indicates that the particular payoff in that entry is not relevant. The notation (x, y, z) indicates that player 1's payoff is x, player 2's payoff is y and player 3's payoff is z.

(a) Find a pure strategy Nash equilibrium that is not subgame perfect. Why is it not subgame perfect?

(b) Find a pure strategy subgame perfect equilibrium that is not a sequential equilibrium. Why is it not a sequential equilibrium?

(c) Find a sequential equilibrium.