[SP26] ECN 812B Recitation 7

Willy Chen

March 13, 2025

1 Concepts this Week

- Repeated games: A stage game (players, strategy sets, **AND** payoffs) that is repeated finitely/infinitely many times
- When there are multiple NEs in the stage game, a non-NE strategy profile is potentially sustainable. i.e., playing non-NE strategies can be a best response when you calculate the total payoff.
  - Recall that strategies are specified combinations of actions at each *information set (not node)*, so to make a strategy in a repeated game setting well-defined, we need to specify the history of the game as part of the strategy.
- Nash Reversion Strategy: A strategy that rewards cooperation with future cooperation, and punishes non-cooperation with NE strategies that are bad for the opponent for the remainder of the game.
- If not using NRS, we need to check the *one-shot deviation principle* to make sure that the punishment makes sense. For example, can playing a non-NE as punishment be a sustainable equilibrium strategy?
- **Principle of Optimality (O&R Lemma 153.1):** A strategy profile is a SPNE of the δ-discounted infinitely repeated game of stage game Γ *if and only if* no player can gain by deviating in a single period after any history. Always check deviation from collaboration, and then make sure to check deviation from punishment.
- **Folk Theorem**: As $\delta \to 1$, any payoff v that is in the convex hull V of the stage game payoffs is feasible as an average payoff in a repeated game PS-SPNE if

  $$v_i > \min_{s_{-i}}\{\max_{s_i} u_i(s_i, s_{-i})\}$$

  If $v_i = \min_{s_{-i}}\{\max_{s_i} u_i(s_i, s_{-i})\}$, we need to make sure that the player enforcing the punishment does not have an incentive to deviate and not punish. i.e., the punishment cannot be too harsh on the person enforcing it.

- The payoff v described in Folk theorem is the "average payoff" calculated as (let v^t denote payoff in period t):

$$\begin{aligned} v_i + \delta \cdot v_i + \delta^2 \cdot v_i + \cdots &= v_i^1 + \delta \cdot v_i^2 + \delta^2 \cdot v_i^3 + \cdots \\ \iff \frac{v_i}{1-\delta} &= v_i^1 + \delta \cdot v_i^2 + \delta^2 \cdot v_i^3 + \cdots \\ \iff v_i &= (1-\delta)(v_i^1 + \delta \cdot v_i^2 + \delta^2 \cdot v_i^3 + \cdots) \end{aligned}$$

  So whatever you solve as the total expected payoff from the SPNE, the average payoff (used in Folk theorem) is the total payoff multiplied by $(1-\delta)$.

2 Learning by Doing

1. (Modified from MSU Prelim SS 2016 Part II Q1) Let $x \geq 0$, consider the following prisoner's dilemma game repeated $T \in \mathbb{N}$ times with discount rate $\delta \in (0, 1)$.

		P_2	
		C	D
P_1	C	$3, 3$	$1, x$
	D	$5, 1$	$0, 0$

   (a) What is the range of x for which we can find a δ that sustains a non-NE profile being played in every period until the T^{th} period?

   Solution.
   To be able to sustain a non-NE profile, it must be that the stage game has multiple NEs; otherwise, the only SPNE would be to play the unique NE in every stage game. As such, for $x \in [3, \infty)$, we can find a δ that sustains a non-NE profile being played in every period.

   (b) Suppose that the stage game is now repeated infinitely many times. Let x be in the range found in (a). What is the Nash Reversion Strategy that can sustain the profile (C, C) being played in every stage game.

   Solution.
   If x is in the range found in part (a), then this game has 2 PSNEs: (C, D) and (D, C). To sustain (C, C) with Nash Reversion, it must be that the players will punish by deviating to playing D forever starting in period t if their opponent deviates in period $t - 1$. So the Nash Reversion strategy is:

$$s_i^t = \begin{cases} C_i & \text{if } H^{t-1} = (C_i, C_j),\ H^{t-1} = (C_i, D_j),\ \text{or } t = 1 \\ D_i & \text{Otherwise} \end{cases}$$

   where H^{t-1} is the strategy profile observed by both player at the end of period $t - 1$.

(c) Without doing any formal analysis, what is the relationship between δ and x if we want to sustain the profile (C, C) being played in every stage game?

Solution.

As x increases, the instant reward for deviation gets bigger and bigger, meaning that the loss from future punishments needs to be greater to prevent profitable deviation. So x and δ are positively correlated.

(d) Find the smallest δ such that there is a subgame perfect equilibrium of the infinitely repeated game with discounting in which both players play C every period along the equilibrium path. Is your answer in (c) consistent with this result?

Solution.

For P_1, the deviation is not profitable if

$$5 + \delta \cdot 1 + \delta^2 \cdot 1 + \cdots \leq 3 + \delta \cdot 3 + \delta^2 \cdot 3 + \cdots$$
$$\iff 5 + \frac{\delta}{1-\delta} \leq \frac{3}{1-\delta} \iff \delta \geq \frac{1}{2}$$

For P_2, the deviation is not profitable if

$$x + \delta \cdot 1 + \delta^2 \cdot 1 + \cdots \leq 3 + \delta \cdot 3 + \delta^2 \cdot 3 + \cdots$$
$$\iff x + \frac{\delta}{1-\delta} \leq \frac{3}{1-\delta} \iff \delta \geq \frac{x-3}{x-1}$$

The minimum δ that can sustain (C, C) via NRS is $\hat{\delta} = \max\left\{\frac{1}{2}, \frac{x-3}{x-1}\right\}$, meaning

$$\hat{\delta} = \begin{cases} \frac{1}{2} \text{ if } x \in [3, 5] \\ \frac{x-3}{x-1} \text{ if } x \in (5, \infty) \end{cases}$$

So for $x > 5$, we have $\frac{d}{dx}\hat{\delta} = \frac{x-1-(x-3)}{(x-1)^2} = \frac{2}{(x-1)^2} > 0$. So as x increases from 5, the minimum δ needed to sustain (C, C) increases. This is consistent with the intuition in (c).

2. (Modified from MSU Prelim May 2023) Consider a Cournot model with three firms with 0 marginal cost. The inverse demand function in the market is $P(Q) = 12 - Q$. The Cournot outcome is $(q_1, q_2, q_3) = (3, 3, 3)$ with profits $(\pi_1, \pi_2, \pi_3) = (9, 9, 9)$. Let this game be played with infinite repetition, we want to consider the possibility of collusion that sustains the monopoly outcome in the market with repeated interactions. Specifically, construct an SPNE such that each firm produces $\frac{1}{3}$ of the monopoly quantity and use the stage game Nash equilibrium as the punishment strategies (Nash reversion). Find the condition on δ, such that the SPNE can be sustained.

Solution.

[Note that the sollution for this is verbatim from my own prelim answer.]

The monopoly quantity is $q^M \equiv \underset{q}{argmax}\ (12 - q)q$, so the F.O.C. is

$$12 - 2q = 0 \ \Rightarrow \ q^M = 6, \ \pi^M = 36$$

The Nash Reversion is then

$$q_i^t = \begin{cases} q_i = 2 & , H^{t-1} = (2, 2, 2) \text{ or } t = 1 \\ q_i = 3 & , \text{otherwise} \end{cases}$$

If firm j deviates, their BR is

$$q^D \equiv \underset{q}{argmax}\ (12 - 4 - q)q$$

The F.O.C. is $8 - 2q = 0 \ \Rightarrow \ q^D = 4, \pi^D = 16$. So to sustain NRS, we need δ such that

$$12 + \frac{\delta}{1-\delta} \cdot 12 \geq 16 + \frac{\delta}{1-\delta} \cdot 9 \iff \frac{3\delta}{1-\delta} \geq 4$$
$$\iff 3\delta \geq 4 - 4\delta \iff 7\delta \geq 4 \iff \delta \geq \frac{4}{7}$$

By symmetry, same δ for all firms. So for $\delta \in \left[\frac{4}{7}, 1\right)$, this NRS is sustainable.

3. (MSU Prelim FS 2017 Q3) Consider the following buyer-seller game: the buyer (B) decides whether to buy (b) or not (n) whereas the seller (S) decides whether to provide high quality (H) or low quality (L). The decisions are simultaneous. The payoffs are as follows:

		B: b	B: n
S	H	$2, 3$	$0, 0$
	L	$3, 2$	$0, 0$

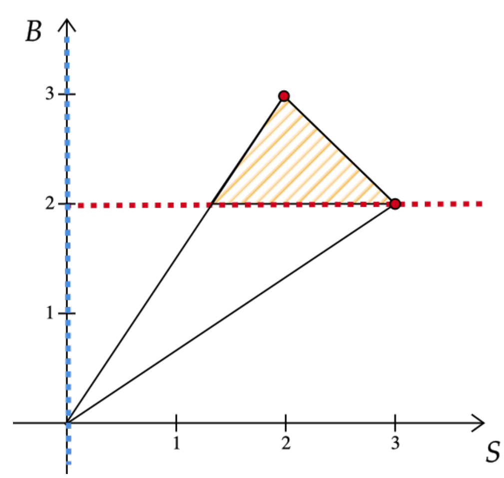
(a) What is the Nash Equilibrium of this game and what are the minmax payoffs of the two players?

Solution.
The unique PSNE is (L, b).
The minmax for S is 0 and the minmax for B is 2.

Next, consider an infinitely repeated game where the above stage game is played in each period. Both parties have a common discount factor $\delta \in (0, 1)$. Answer the following questions:

(b) Sketch the set of feasible payoffs and the set of average payoffs that can be sustained in an SPNE as $\delta \to 1$ (as per the folk-theorem discussed in class).

Solution.



As you may see from part (b) above, folk theorem would imply that there exists an SPNE for δ sufficiently large where in each period the buyer buys and the seller provides high quality. That is, we can have $(2, 3)$ be the of every stage game in SPNE. Consider a set of strategies where (H, b) is played in each period but if the seller "cheats" the buyer by choosing L, the buyer expects the seller to continue to choose L until the buyer punishes him by not buying. Formally, the strategies are as such:

Buyer: In period 1, play b. In any period $t > 1$, play n if the last period's $(t-1)$ play is (L, b). Otherwise, always play b.

Seller: In period 1 play H. In any period $t > 1$, play L if the last period's $(t-1)$ play is (L, b). Otherwise, always play H.

(c) Find the value of minimum δ such that the above strategies constitute an SPNE. [*Hint:* Note that δ has to be high enough so that seller has incentive to choose H and also the buyer has incentive to choose n if seller has deviated in the past.]

Solution.
Given the prescribed strategy profile, the seller will not deviate to L if:

$$3 + \delta \cdot 0 + \delta^2 \cdot 2 + \cdots \leq 2 + \delta \cdot 2\delta^2 \cdot 2 + \cdots \iff \delta \geq \frac{1}{2}$$

And it also must be that buyer's punishment strategy is sustainable (i.e., always buying is not BR):

$$2 + \delta \cdot 0 + \delta^2 \cdot 3 + \cdots \leq 0 + \delta \cdot 3 + \delta^2 \cdot 3 + \cdots \iff 2 + \frac{3\delta^2}{1-\delta} \leq \frac{3\delta}{1-\delta}$$
$$\iff 3\delta^2 - 5\delta + 2 \leq 0 \iff (3\delta - 2)(\delta - 1) \leq 0 \iff \delta \geq \frac{2}{3}$$

So the minimum δ to sustain this strategy profile as SPNE is $\frac{2}{3}$.

4. (Columbia Prelim 2011) Recall that the stage game of Matching Pennies is characterized by the following payoff matrix:

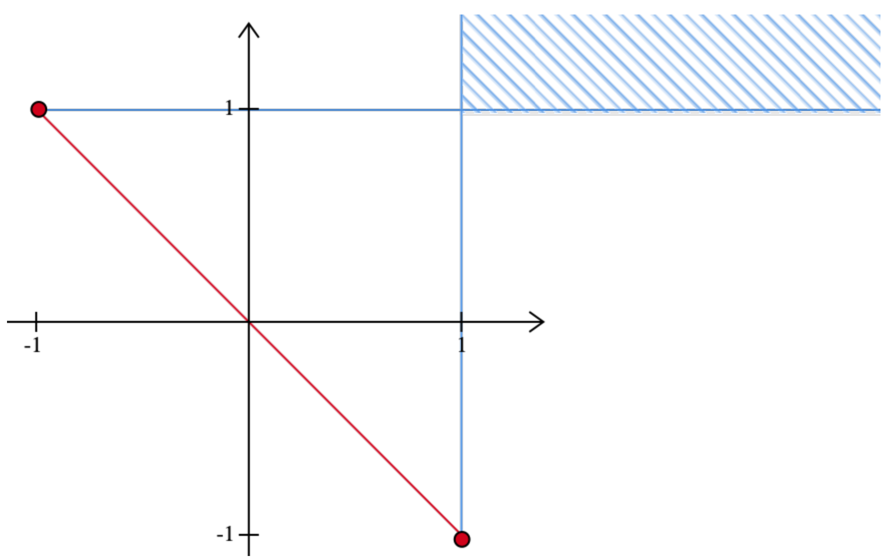
P_1 \ P_2	Heads	Tails
Heads	$-1, 1$	$1, -1$
Tails	$1, -1$	$-1, 1$

(a) What does the folk theorem imply for the infinitely repeated Matching Pennies game? Explain your answer fully.

Solution.
The convex hull of the payoffs is the straight line connecting the points $(1, -1)$ and $(-1, 1)$. The minmax for player i is:

$$\min_{s_j}\{\max_{s_i}\{u_i(H,H), u_i(H,T)\}, \max_{s_i}\{u_i(T,H), u_i(T,T)\}\} = \min\{1,1\} = 1$$

The set of feasible payoff is the part of the straight (red) line that overlaps with the (blue) patterned area. Since they do not overlap, Folk theorem implies that there is no PS-SPNE in the infinitely repeated matching pennies game.



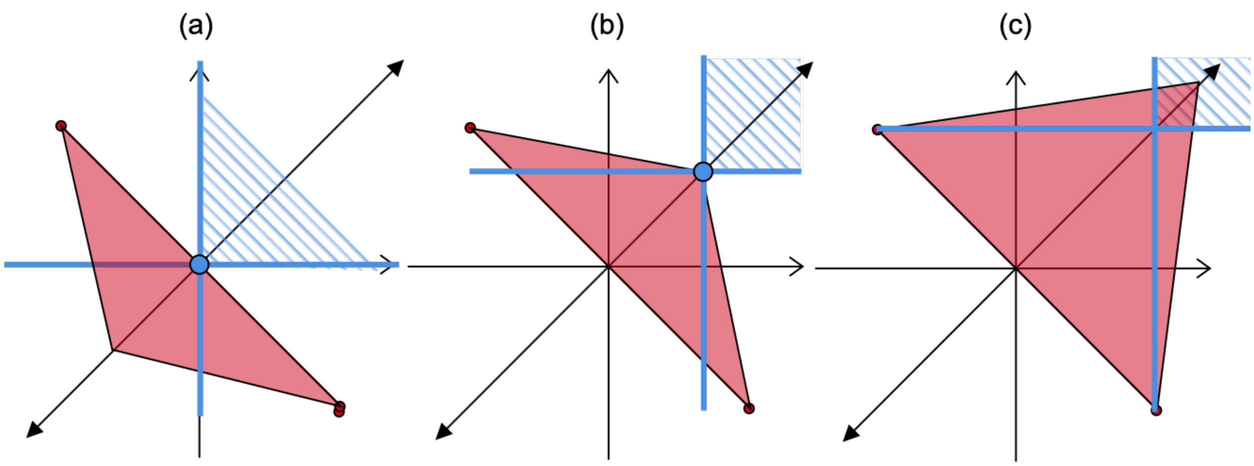
(b) How would your answer change if the stage-game is changed as follows:

$P_1 \backslash P_2$	Heads	Tails	Other
Heads	$-1, 1$	$1, -1$	$0, 0$
Tails	$1, -1$	$-1, 1$	$0, 0$
Other	$0, 0$	$0, 0$	x, x

Solution.

Notice that under this new game, the minmax of player i changes to $\min\{1, \max\{x, 0\}\}$.

- (a) If $x < 0$, then only $(0, 0)$ is sustainable in PS-SPNE.
- (b) If $0 \leq x \leq 1$, then only (x, x) is sustainable in PS-SPNE.
- (c) If $1 \leq x$, then part of the convex hull is sustainable in PS-SPNE.



3 Go the Extra Mile

1. (2019 Practice Exam Q3) The simultaneous-move game below is played twice, with the outcome of the first stage observed before the second stage begins. There is no discounting. Can the payoff $(4,4)$ be achieved in the first stage in a pure-strategy subgame-perfect Nash equilibrium? If so, give strategies that do so. If not, prove why not.

Player 1 \ Player 2	L	C	R
T	$3,1$	$0,0$	$5,0$
M	$2,1$	$1,2$	$3,1$
B	$1,2$	$0,1$	$4,4$

2. (MSU Prelim FS 2014, Part II Q2) Consider the following stage games:

($Game$ 1)

P_1 \ P_2	C	D
C	$4,4$	$0,5$
D	$6,0$	$1,1$

($Game$ 2)

P_1 \ P_2	C	D
C	$4,4$	$0,6$
D	$5,0$	$1,1$

(a) Consider Game 1. Find the Nash equilibrium and minmax payoff. In a labeled diagram, sketch the set of individually rational payoffs.

(b) Suppose two players play only Game 1 (or only Game 2) infinitely many times, with common discount factor and perfect monitoring. What is the minimum δ that will allow the existence of a subgame perfect Nash equilibrium in which the players play $(C;C)$ along the equilibrium path (consider grim-trigger strategies)?

(c) Now suppose the two players will play both games simultaneously in each period. Find the minimum δ that will allow the existence of a subgame perfect Nash equilibrium in which they play $(C;C)$ in both games along the equilibrium path.

(d) When is it easier to sustain cooperation—when they play Game 1 only or when they play both games simultaneously? Provide a brief intuition for your result.

3. (MSU 2021 Exam 2 Q2) Consider a game with two players ($i = 1, 2$) with the following action and payoff space:

		P_2: L	C	R
P_1	U	$3, 3$	$0, 4$	$-\frac{3}{2}, -1$
	M	$4, 0$	$1, 1$	$-1, -1$
	D	$-1, -\frac{3}{2}$	$-1, -1$	$-2, -2$

(a) Suppose that the two players move sequentially. Player 1 moves first with a publicly observable action, followed by Player 2. Find all pure strategy SPNE of this sequential game.

(b) Suppose now that the two players move simultaneously. They play the simultaneous move game twice and both possess a common discount factor $\delta \in (0, 1)$. Find all pure strategy SPNE of this repeated game.

(c) Use the Folk Theorem to characterize and/or draw the set of feasible average discounted payoffs in an SPNE, clearly identifying the minmax strategies and payoffs.

(d) Suppose now that the game is repeated infinitely. Define a grim trigger strategy for each player such that (U, L) is the outcome of the infinitely repeated game in each period, with a threat of minmax punishment. What is the lowest δ such that (U, L) is realized every period?

(e) Is the strategy you specified in the previous part an SPNE? Why or why not? If not, specify another grim trigger strategy with (U, L) as an SPNE outcome of the infinitely repeated game. What is the lowest δ such that this strategy can be supported as an equilibrium?

4. (MSU 2016 Final Q3) Consider the following stage game with three players all having action sets $\{A, B\}$. Suppose P_3 chooses the matrix:

A:

		P_2: A	B
P_1	A	1,1,1	0,0,0
	B	0,0,0	0,0,0

B:

		P_2: A	B
P_1	A	0,0,0	0,0,0
	B	0,0,0	1,1,1

Suppose the above stage game is repeated infinitely many times. The discount factor is $\delta \in (0, 1)$ for all players. Can there exist a SPNE of this game where a player's average payoff is in $[0, \frac{1}{4})$? If so, state the strategy profiles that sustain such a payoff in SPNE. If not, give a proof. [*Hint*: No matter what strategy profile is chosen, note that in any period at least two players would play the same action with at least 1/2 probability. Using this fact, try to deduce what could be the minimum payoff a player can get in a SPNE.]

Solution.

For any SPNE strategy profile, suppose the play in period 1 calls for i playing A with probability $\sigma_i(A)$. Now there must be at least two players, say i and j such that either $\sigma_i(A), \sigma_j(A) \geq 1/2$ or $\sigma_i(B), \sigma_j(B) \geq 1/2$. (This is same as saying that in toss of three coins, we must have at least two heads or at least two tails). Suppose without loss of generality, player 1 and 2 play A with at least 1/2 probability. Now P_3 can guarantee $\frac{1}{4}$ by playing A as well. Let x be the infimum of all SPNE payoff of player 3. So, if this equilibrium gives P_3 an average payoff of v, we must have:

$$v \geq (1-\delta)\frac{1}{4} + \delta x.$$

Since the above inequality holds for any v that could be supported in SPNE, we must have:

$$x = \inf_{v \ is \ SPNE \ payoff} v \geq (1-\delta)\frac{1}{4} + \delta x \ or \ x \geq \frac{1}{4}.$$

Hence, there does not exist any SPNE where a player's average payoff is less than $\frac{1}{4}$.