[SP26] ECN 812B Recitation 8

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1 Concepts this Week

- A **Bayesian Game** is a game $\Gamma_N = \{\mathcal{I}, \{S_i\}, \{u_i\}, \Theta, F\}$ where Θ is the collection of player types and F is the distribution of types.
- A **Reformulated Bayesian Game** is a game $\Gamma_N = \left\{\mathcal{I}, \left\{\prod_{\theta_i} S_i^{\theta_i}\right\}, \{\bar{u}_i\}\right\}$
- BNEs are just NEs where strategies are specified for each **player-type combination**, and not just for each player.
- Continuous Type BNE: It's the same thing as discrete type, always remember to specify your strategies to be contingent on each agent's own known types.
  - In auctions, the strategies are bids as a function of own/known valuations. When you check for deviations, differentiate on the bid b_i and not the strategy $b(v_i)$. One common mistake is to take the derivative of $b(v_i)$ with respect to v_i.
  - Bidding strategies often implicitly assume monotonicity, i.e., $\frac{d}{dv_i} b(v_i) > 0$. Sometimes this is not given and you may need to check for that.
  - The BNE maximizes one's own expected utility given own valuation v_i.
- When solving continuous type BNE problems where a specific realization of type makes the player indifferent, you can arbitrarily specify their strategy for that specific point since it happens with probability 0 (This is often not true with discrete types).

*Recitation is taught by Abhay Gupta this week.

2 Learning by Doing

1. (Modified from MSU Prelim FS 2013 Part II Q4) The water cooler in Old Botany has run out of water. Two econ graduate students S_1, S_2 simultaneously decide whether to go to Marshall-Adams to pick up a jug of water, which costs them in effort a_i, $i \in \{1, 2\}$. As long as one student picks up the water, there will be water in Old Botany and both would receive a payoff of 4 (Assume that they only derive payoff from satisfying their immediate needs for water). If both grad students pick up the water, then the extra jug is redundant. Consequently, the grad students have two pure strategies: pick up water (P) or free-ride (F). The payoff table is:

$S_1 \setminus S_2$	P	F
P	$4-a_1, 4-a_2$	$4-a_1, 4$
F	$4, 4-a_2$	$0, 0$

   (a) Find all Nash equilibria if $a_1 = a_2 = 3$. In the mixed strategy equilibrium, what is the probability that there will be water in the Old Botany water cooler?

   Solution.
   In this case, the payoff table is

$S_1 \setminus S_2$	P	F
P	$1, 1$	$\underline{1}, \underline{4}$
F	$\underline{4}, \underline{1}$	$0, 0$

   There are 2 PSNEs: (P_1, F_2) and (F_1, P_2).
   There is also an MSNE $\left(\frac{1}{4}P_1 + \frac{3}{4}F_1, \frac{1}{4}P_2 + \frac{3}{4}F_2\right)$.
   In this MSNE, the probability that someone grab the water is $1 - \frac{9}{16} = \frac{7}{16}$.

(b) Now suppose that a_i is private information of each individual, with it being common knowledge that $Pr(a_i = 1) = Pr(a_i = 3) = \frac{1}{2}$. Prove or disprove that the game has a Bayesian Nash equilibrium in which low-cost grad students pick up a jug and high-cost grad students free ride.

Solution.

Suppose that such BNE exists. For the low-cost grad students, their expected utility for picking up water versus free riding is:

$$u_L(E) = \frac{1}{2} \cdot (4-1) + \frac{1}{2} \cdot (4-1) = 3 > 2 = \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 0 = u_L(F)$$

For the high-cost grad students, their expected utility for picking up water versus free-riding is:

$$u_H(E) = \frac{1}{2} \cdot (4-3) + \frac{1}{2} \cdot (4-3) = 1 < 2 = \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 0 = u_H(F)$$

Given this strategy profile, the low-cost grad students have no incentive to free ride, and the high-cost grad students have no incentive to pick up water. As such, the strategy profile where only low-cost grad students pick up water and only high-cost grad students free ride is a BNE.

2. (Modified from Harvard Prelim 2016) Consider the following 2-player game of incomplete information:

P1 \ P2	T_UT_D	T_UB_D	B_UT_D	B_UB_D
U	$2, 3+V$	$2, 3+V$	$-1, 4-V$	$-1, 4-V$
D	$3, -1$	$0, 0$	$3, -1$	$0, 0$

(a) Suppose that $V \in \{0, 1\}$, P_2 knows the realization of V but P_1 does not and they believe that $Pr(V = 1) = p$. Find all pure-strategy Bayes-Nash equilibria of this game.

Solution.

Notice that if $V = 1$, P_2 would prefer T to B when P_1 plays U. If $V = 0$, P_2 would prefer B to T when P_1 plays U. They also always prefer B to T if P_1 plays D.

This means that, from P_1's perspective, P_2 is playing a mix between T_UB_D and B_UB_D where they play the former with probability p. So P_1's expected payoff of playing U is $2p-(1-p) = 3p-1$ and their expected payoff when playing D is 0.

So the PS-BNEs are as follows:

- For $p > \frac{1}{3}$, $(s_1, s_2(V = 0), s_2(V = 1)) = (U, B_UB_D, T_UB_D)$.
- For $p = \frac{1}{3}$, $(s_1, s_2(V = 0), s_2(V = 1)) = (U, B_UB_D, T_UB_D)$.
- For $p \leq \frac{1}{3}$, $(s_1, s_2(V = 0), s_2(V = 1)) = (D, B_UB_D, T_UB_D)$.

(b) Suppose that $V \sim U[0, x]$ and the rest of the game is the same. Find the appropriate x such that P_1 plays a completely mixed strategy in the Bayes-Nash equilibrium of this game. Write out the BNE.

Solution.

Notice that P_2 only prefers $T_U B_D$ to $B_U B_D$ if $3 + V \geq 4 - V \Rightarrow V \geq \frac{1}{2}$, which happens with probability $\frac{x - \frac{1}{2}}{x} = 1 - \frac{1}{2x}$.

From (a), we know that P_1 is only indifferent between U and D if $p = \frac{1}{3}$, so we solve:

$$1 - \frac{1}{2x} = \frac{1}{3} \Rightarrow \frac{1}{2x} = \frac{2}{3} \Rightarrow x = \frac{3}{4}.$$

As such, the BNE is:

$$s_1 = qU + (1-q)U, q \in [0,1]; s_2 = \begin{cases} T_U B_D, & V \in [\frac{1}{2}, \frac{3}{4}] \\ B_U B_D, & V \in [0, \frac{1}{2}] \end{cases}$$

3. (UChicago Prelim 2012) Players 1 and 2 must independently decide whether to grab or pass. Each player i privately knows their payoff t_i from yielding, which we may call the player's type. For a given pair (t_1, t_2), the game is described by the following table:

		P_2	
		Grab	Pass
P_1	Grab	$-2, -2$	$8, t_2$
	Pass	$t_1, 8$	t_1, t_2

(a) Suppose that $t_1 = 0$ and $t_2 \in \{0, 1\}$ with $Pr(t_2 = 1) = \frac{1}{2}$. Find a Bayes-Nash equilibrium in which player 1 plays a completely mixed strategy.

Solution.

In BNE, if P_1 plays a completely mixed strategy, it must be that they are indifferent between grab and pass. As such, it must be that P_1 expects P_2 to grab with probability p such that:

$$-2p + 8(1-p) = 0 \;\Rightarrow\; p = \frac{4}{5}$$

Notice that when $t_2 = 0$, P_2 would be indifferent between grab and pass if P_1 grabs with probability $\frac{4}{5}$. Similarly, when $t_2 = 1$, P_2 would be indifferent between grab and pass if P_1 grabs with probability $\frac{7}{10}$. Since P_2 needs to play $p = \frac{4}{5}$ for P_1 to be indifferent and $Pr(t_2 = 1) = \frac{1}{2}$, it must be that P_2 grabs with probability 1 when $t_2 = c$ and with probability q when $t_2 = 1 - c$.

If P_1 grabs with probability $\frac{4}{5}$ and $t_2 = 1$, then P_2 would prefer to pass.
If P_1 grabs with probability $\frac{7}{10}$ and $t_2 = 0$, then P_2 would prefer to grab.

As such, the BNE here is

$$\sigma_1(G_1, P_1) = \left(\frac{7}{10}, \frac{3}{10}\right) \;,\; \sigma_2(G_2, P_2 \mid t_2) = \begin{cases} (1, 0) & , t_2 = 0 \\ (\frac{3}{5}, \frac{2}{5}) & , t_2 = 1 \end{cases}$$

(b) Suppose that t_1 and t_2 are independent random variables each drawn from $U[0,1]$, and each player only knows their realized type. Find a Bayes-Nash equilibrium in which, for each player, some types would grab but other types would yield.

Solution.

Suppose that such an equilibrium exists, then it must be that $\exists \bar{t} \in [0,1]$ such that P_i would grab with probability 1 if $t_i < \bar{t}$, meaning:

$$E[u_i(G_i, \sigma_j) \mid t_i < \bar{t}\,] > E[u_t(P_i, \sigma_j) \mid t_i > \bar{t}\,]$$

By symmetry, in equilibrium, P_j must also play the same strategy, meaning that in P_i's view, P_j plays grab with probability $P(t_j < \bar{t}) = \bar{t}$. We can calculate $\bar{t}$ by capturing when P_i would be indifferent between G_i and P_2, meaning:

$$E[u_i(G_i, \sigma_j) \mid t_i = \bar{t}\,] = E[u_t(P_i, \sigma_j) \mid t_i = \bar{t}\,]$$

Given P_j's strategy, this equality can be re-written as:

$$\underbrace{-2 \cdot P(t_j < \bar{t})}_{-2\bar{t}} + \underbrace{8 \cdot P(t_j > \bar{t})}_{8(1-\bar{t})} = \bar{t} \;\Rightarrow\; \bar{t} = \frac{8}{11}$$

As such, the BNE is:

$$s_1 = s_2 = s_i = \begin{cases} Grab & , t_i < \frac{8}{11} \\ Pass & , t_i \geq \frac{8}{11} \end{cases}$$

4. (MSU Prelim SS 2018 Q3) Suppose that two bidders are bidding for an object that has either high or low quality, or equivalently, value $v \in \{0, V\}$ to both of them. However, neither of the two bidder knows the value for sure, and the outcomes are equally likely. But they both receive a private signal $s_i \in \{l, h\}$ that is informative of the value, as

$$Pr(v = V \mid s_i = l \text{ for some } i) = 0 \text{ and } Pr(v = V \mid s_1 = s_2 = h) = p > \frac{1}{2}$$

The auction format is "first-price sealed bid": object goes to the highest bidder and she pays what she has bid. Loser does not pay anything. Notice that if a symmetric pure strategy BNE exists for this game, both players must bid according to some function $b(s_i)$ where $b(s_i = l) = b_l$ and $b(s_i = h) = b_h \geq b_l$

(a) Suppose a symmetric pure BNE exists. What is the value of b_l in the equilibrium?

Solution.
Clearly, if $s_i = l$, i knows $v = 0$, so $b_l = 0$ in equilibrium.

(b) Using your answer to the question above, either derive a symmetric pure strategy BNE, or prove that none exists.
Solution.
Suppose there is a symmetric PS-BNE. Since player i knows that $P(v = 0) = P(v = V) = \frac{1}{2}$, if they receive $s_i = h$ and bid b_h, they either win with probability $\frac{1}{2}$ when $s_j = h$ (because player j would only bid b_h if they also observed h.) or win with probability 1 when $s_j = l$. So their expected payoff for bidding b_h when observing $s_i = h$ is:

$$\frac{1}{2} \cdot \underbrace{P(s_j = h \mid s_i = h)}_{\text{Call this } q} \cdot (pV - b_h) + P(s_j = l \mid s_i = h)(-b_h) = \frac{q(pV - b_h)}{2} - (1 - q)b_h$$

If $b_h < qpV$, then player i would have an incentive to deviate by playing $b_h + \varepsilon$ and increase their expected payoff

$$q(pV - b_h - \varepsilon) - (1 - q)(b_h - \varepsilon) = q(pV - b_h) - (1 - q)b_h - \varepsilon$$

If $b_h \geq qpV$, then player I would have negative expected payoff, meaning they would have an incentive to deviate to 0. As such, $b(s_i = h) = b_h$ is not a symmetric PS-BNE.

5. (MSU Final 2024) Consider a first-price sealed-bid auction of a single indivisible good with two bidders. Each bidder values the good at either v_L or v_H. The bidders know their own valuation of the good but not the other bidder's valuation. The common prior is that $P(v_i = v_L) = \lambda$. In case of a tie, a fair two-sided coin is flipped to decide the winner of the auction.

   (a) Show that there is no symmetric pure strategy Bayes-Nash equilibrium in this game.

   Solution.
   Suppose that a symmetric equilibrium bidding strategy exists such that $b(v_H) = b_H$ and $b(v_L) = b_L$ such that $b_H > b_L$. First, notice that it must be that $b_L = v_L$. If $b_L < v_L$, the bidders with valuation always has an incentive to bid $b_L = v_L - \varepsilon$ for sufficiently small ε and get payoff $\lambda\varepsilon > 0$.

   Suppose that $b_H < v_H$. For bidders with valuation v_H, their expected payoff for following this strategy is:

$$(1-\lambda)\frac{v_H - b_H}{2} + \lambda(v_H - b_H).$$

   But if they bid $b'_H = b_H + \varepsilon$, for some small ε, instead, their payoff is $v_H - b'_H$. In equilibrium, this must mean:

$$(1-\lambda)\frac{v_H - b_H}{2} + \lambda(v_H - b_H) = v_H - b_h - \varepsilon \Rightarrow \varepsilon = \frac{(1-\lambda)(v_H - b_H)}{2}.$$

   Since b_H is an equilibrium pure strategy, it must mean that no deviation with $\varepsilon \geq 0$ is profitable, so it must be that $b_H = v_H$. However, if $b_H = v_H$, the bidder's expected payoff is 0, meaning they can have profitable deviation by bidding $v_L + \varepsilon < v_H$ and get expected payoff $\lambda(v_H - v_L - \varepsilon)$. In this case, $b_H < v_H$, so there is profitable deviation to bid $b'_H > b_H$ again.

   As such, there is no symmetric pure strategy BNE in this game.

Suppose that, in equilibrium, a bidder of value v_L still bids $b_L = v_L$. Answer parts (b) through (d) to find the symmetric optimal bidding strategy of bidders with v_H.

(b) What is the maximization problem of a bidder of value v_H?

Solution.

$$\pi_i(b_i) = (v_H - b_i)\left[(1-\lambda)P(b_i > b_j) + \lambda\right]$$

So the maximization problem is:

$$\max_{b_i} (v_H - b_i)\left[(1-\lambda)F(b_i) + \lambda\right]$$

(c) Show that bidder i of value v_H is a mixed strategy following the distribution with cdf $F(b_i)$, where $F(b_i)$ satisfies the equality

$$-\int_{v_L}^{b} \frac{d}{db_i}\left[(v_H - b_i)(1 - F(b_i))\right] db_i = \int_{v_L}^{b} \frac{1}{1-\lambda}\, db_i.$$

Solution.

Taking the F.O.C. with respect to b_i of the maximization problem from (b)

$$\begin{aligned}
(v_H - b_i)[(1-\lambda)f(b_i)] - [(1-\lambda)F(b_i) + \lambda] &= 0 \\
\iff (v_H - b_i)f(b_i) - F(b_i) &= \frac{\lambda}{1-\lambda} \\
\iff 1 - F(b_i) + (v_H - b_i)f(b_i) &= \frac{1}{1-\lambda} \\
\iff -\frac{d}{db_i}\left[(v_H - b_i)(1 - F(b_i))\right] &= \frac{1}{1-\lambda} \\
\iff -\int_{v_L}^{b} \frac{d}{db_i}\left[(v_H - b_i)(1 - F(b_i))\right] db_i &= \int_{v_L}^{b} \frac{1}{1-\lambda} db_i
\end{aligned}$$

(d) Solve for $F(b)$ and argue that the support of b is $[v_L, \mu]$ where $\mu \in (v_L, v_H)$. What happens to μ as λ (the probability that the other bidder is type v_L) decreases?

Solution.

From part (b), since $b_L = v_L$, it must be that $b_H > v_L$.

$$-\int_{v_L}^{b} [(v_H - b_i)(1 - F(b_i))]' db_i = \int_{v_L}^{b} \frac{1}{1-\lambda} db_i$$

$$-[(v_H - b_i)(1 - F(b_i))]\Big|_{v_L}^{b} = \frac{1}{1-\lambda} b_i \Big|_{v_L}^{b}$$

$$-[(v_H - b)(1 - F(b))] + [(v_H - v_L)(1 - F(v_L))] = \frac{1}{1-\lambda}(b - v_L)$$

$$\Rightarrow F(b) = \frac{\lambda}{1-\lambda} \frac{b - v_L}{v_H - b}$$

Therefore, the high type mixes his bid which follows the CDF of $F(b) = \frac{\lambda}{1-\lambda} \frac{b-v_L}{v_H-b}$. | Clearly, $F(b)$ is undefined at v_H, meaning the support of this CDF must be $[v_L, \mu]$ where $\mu \in (v_L, v_H)$. Specifically, we know that:

$$F(\mu) = 1 = \frac{\lambda}{1-\lambda} \frac{\mu - v_L}{v_H - \mu} \iff (1-\lambda)(v_H - \mu) = \lambda(\mu - v_L)$$

$$\iff \mu = (1-\lambda)v_H + \lambda v_L \Rightarrow \frac{d}{d\lambda}\mu = -v_H + v_L < 0$$

As λ decreases, μ increases. This means that if the probability that the other bidder's value is v_H increases, the maximum amount a v_H bidder needs to randomize will also increase, to maximize their expected payoff.

3 Go the Extra Mile

1. Consider the sealed-bid, all-pay auction for a single item, in which bids are selected from the continuum. There are two risk-neutral bidders, whose utilities are quasi-linear in money. Bidder i's valuation, denoted v_i , for the item is private information and the $v_i \overset{iid}{\sim} U[1,2]$. After observing her own valuation v_i, each bidder i simultaneously and independently submits a sealed bid $b_i \in [0,2]$ for the item. The highest bidder wins the item. However, since this is an all-pay auction, every bidder must pay the amount of her bid, regardless of whether she wins or loses. For example, if $b_j > b_i$ then bidder j wins the item; but bidder i pays b_i and bidder j pays b_j . Ties are resolved by a fair randomization. Solve for a symmetric, Bayesian-Nash equilibrium of this game. Explain and justify your work.

   Solution.

   Suppose that the players' bidding strategies are identical strictly increasing function $b(v_i)$ in valuation. Player i solves the problem

$$\max_{b_i} P(b_i > b_j)v_i - b_i \equiv P(b^{-1}(b_i) > v_j)v_i - b_i$$

   The first order condition with respect to b_i is

$$\underbrace{\frac{\partial P(b^{-1}(b_i) > v_j)}{\partial b^{-1}(b_i)}}_{\text{Derivative of CDF=PDF}} \underbrace{\frac{\partial b^{-1}(b_i)}{\partial b_i}}_{=\frac{1}{db/dv}} v_i - 1 = 0 \;\Rightarrow\; 1 \cdot \frac{1}{b'} v_i = 1 \;\Rightarrow\; b = \frac{1}{2}v_i^2 + C$$

   Now, if $v_i = 1$, the expected payoff is 0 (because probability of winning is 0), so the bid should also be 0. As such, we must have

$$\frac{1}{2}1^2 + C = 0 \;\Rightarrow\; C = -\frac{1}{2}$$

   So the BNE bidding function is:

$$b(v_i) = \frac{1}{2}v_i^2 - \frac{1}{2}$$

2. (MSU Prelim FS 2020) A seller plans to sell a painting to one of five potential buyers through a "third-price" auction. The auction environment is defined as follows:

   - It is **common knowledge** that buyer i's valuation is v_i for each $i \in \{1, 2, 3, 4, 5\}$, and that $v_1 > v_2 > v_3 > v_4 > v_5$. That is, buyer 1 values the painting the most (at v_1), followed by buyer 2, and so on.
   - The buyers submit their bids (denoted by $\{b_i\}_{i=1}^5$) **simultaneously** in sealed envelopes.
   - The buyer who submits the highest bid wins the object and pays the **third highest** bid.
   - The buyers who lose the auction do **not** pay anything.
   - If more than one buyer submit the highest bid, the object is randomly allocated to the highest bidders.

   Answer the following questions:

   (a) Show that for any buyer, bidding less than his/her valuation is weakly **dominated** by bidding his/her true valuation (i.e., bidding $b_i < v_i$ is weakly dominated by bidding $b_i = v_i$).

   (b) Show that it is **not** a weakly **dominant** strategy for a bidder to bid his/her true valuation.

   (c) Does there exist a Nash Equilibrium of the bidding game where every buyer bids his/her true valuation, i.e., $b_i = v_i$, $\forall i$? Justify your answer.

   (d) Find **a** pure strategy Nash equilibrium in which the object is **awarded to buyer 1** (the one with the highest valuation, v_1).

3. (Modified from Columbia Prelim 2010) Consider the following game of incomplete information between an economics first-year student (S) and their TA Willy (W). These two agents simultaneously decide on either to spend little time (L) on final prep or to spend much time (M) on final prep. The final is either easier (E) than the qualifying exam (with probability p) or similar (H) to the qualifying exam (with probability $1-p$). Willy knows the type of the exam, but he cannot disclose that information. Their payoffs are described by the following payoff table:

		$E-W$		$H-W$	
		L	M	L	M
S	L	$0,0$	$7,-2$	$-2,-2$	$5,0$
	M	$-2,7$	$5,5$	$0,5$	$7,7$

Find all the Bayes-Nash equilibria in the game.

Solution.

The reformulated Bayesian game payoff table is:

		W			
		LL	LM	ML	MM
S	L	$2p-2, 2p-2$	$5-5p, 0$	$9p-2, -2$	$2p+5, -2p$
	M	$-2p, 2p+5$	$7-9p, 7$	$5p, 5$	$7-2p, 7-2p$

From the original table, we know that if the exam is type E, Willy's dominant strategy is L, and if the exam is type H, Willy's dominant strategy is M. So it must be that LM is the dominant strategy for Willy in the reformulated Bayesian game. Knowing that, the student S's best response would be:

$$BR(LM) = \begin{cases} L & , p > \frac{1}{2} \\ M & , p < \frac{1}{2} \\ qL + (1-q)M,\ q \in [0,1] & , p = \frac{1}{2} \end{cases}$$

The BNEs in this game are thus $(BR(LM \mid p), LM)$.

4. (UChicago 302 PS7) N firms are competing in a single market. Each firm's exact constant marginal cost is private information. It is common knowledge that each firm i's marginal cost, c_i is drawn independently from a $Uniform[0,1]$ distribution. There are no fixed costs. Given its marginal cost, c_i, each firm i must simultaneously choose a quantity, $q_i \in [0,\infty)$ to produce. If the total quantity produced by the N firms is Q the market-clearing price that each firm receives for each unit produced is $P = \alpha - \beta Q$ where $\alpha > 1$ and $\beta > 0$. Each firm seeks to maximize it's expected profits.

   (a) Formulate this situation as a Bayesian game.

   Solution.
   The Bayesian game is $\Gamma_B = \{\mathcal{I}, \{S_i^{c_i)}\}, U_i^{c_i}, \Theta, F(c_i) = c_i\}$ where

   - $\mathcal{I} = \{1, \dots, N\}$
   - $S_i^{c_i} = [0,\infty)$
   - $\forall c_i \in \Theta$, $c_i \overset{iid}{\sim} U[0,1]$
   - $U_i(q_i, q_{-i} \mid c_i) = q_i \left(\alpha - \beta \sum_{j \neq i}^{N} E[q_j(c_j) - \beta q_i] - c_i \right)$

(b) Find a pure strategy Bayes-Nash equilibrium.

(i) Find c^* such that $q_i > 0$ if $c_i \leq c^*$ and $q_i = 0$ if $c_i > c^*$

Solution.

Suppose that every firm plays this prescribed strategy: $q_i = q(c_i)\mathbb{1}\{c_i \leq c^*\}$. Then each firm is expected to produce $q^e = \int_0^{c^*} q(c)dF(c)$. As such, if $c_i < c^*$, the expected profit of firm i is:

$$E[\pi_i \mid x_i] = q_i \left(\alpha - \beta(N-1)q^e - \beta q_i - c_i\right)$$

So firm i's best response is:

$$\begin{aligned} \frac{\partial}{\partial q} &= \alpha - \beta(N-1)q^e - 2\beta q - c_i = 0 \\ \iff q(c_i) &= \frac{\alpha - \beta(N-1)q^e - c_i}{2\beta} \end{aligned}$$

By the definition of c^*, it must be that $c^* = \alpha - \beta(N-1)q^e$, so $q(c_i) = \frac{c^* - c_i}{2\beta}$ and so we know

$$\begin{aligned} q^e &= \int_0^{c^*} \frac{c^* - c}{2\beta} dc = \frac{c^{*2} - \frac{1}{2}c^{*2}}{2\beta} = \frac{c^{*2}}{4\beta} \\ \Rightarrow\ c^* &= \alpha - \beta(N-1)\frac{c^{*2}}{4\beta} \\ \Rightarrow\ 0 &= \frac{N-1}{4}c^{*2} + c^* - \alpha \\ \Rightarrow\ c^* &= \frac{2\left(-1 + \sqrt{1 + \alpha(N-1)}\right)}{N-1} > 0 \end{aligned}$$

(ii) What happens to each firm's output, as a function of its marginal cost, as the number of firm's tends to infinity? What happens to total output and what's the market clearing price?

Solution.

From (b)(i), we know that the cutoff value $c^* \to 0$ as $N \to \infty$, so each firm would individually produce approximately zero. However, for total outputs, the quantity is calculated as $(N-1) \cdot q^e$ where

$$q^e = \frac{c^{*2}}{4\beta} = \frac{4\left(-1+\sqrt{1+\alpha(N-1)}\right)^2}{4\beta(N-1)^2} \to 0 \text{ as } N \to \infty$$

. As such, as $N \to \infty$, we have total quantity:

$$\lim_{N\to\infty} Nq^e = \lim_{N\to\infty} \left(\frac{N(2+\alpha(N-1)-2\sqrt{1+\alpha(N-1)})}{\beta(N-1)^2} \right) = \frac{\alpha}{\beta}$$

The market clearing price is thus

$$\alpha - \beta \cdot \frac{\alpha}{\beta} = 0$$

as $N \to \infty$