[SP26] ECN 812B Recitation 9

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1 Concepts this Week

- Continuous Type BNE: It's the same thing as discrete type, always remember to specify your strategies to be contingent on each agent's own known types.
  - In auctions, the strategies are bids as a function of own/known valuations. When you check for deviations, differentiate on the bid b_i and not the strategy $b(v_i)$. One common mistake is to take the derivative of $b(v_i)$ with respect to v_i.
  - Bidding strategies often implicitly assume monotonicity, i.e., $\frac{d}{dv_i}b(v_i) > 0$. Sometimes this is not given and you may need to check for that.
  - The BNE maximizes one's own expected utility given own valuation v_i.
- Adverse Selection (Preview): The phenomenon where asymmetric information leads to inefficiency, and sometimes the complete "unravelling" of the market.
  - Instead of making decisions using the expected value of a good, buyers/sellers make decisions using the *conditional expected value* of a good.
  - For example, if the buyers only know a good is valued at $v \sim U[0,1]$ but seller knows thee exact value and will only sell if $p \geq v$. Then the expected value of the good priced at v is $E[v \mid v \leq p] = \frac{p}{2}$ instead of $E[v] = \frac{1}{2}$.

2 Learning by Doing

1. (Modified MSU May 2022) A seller wants to sell a single object and has two interested buyers. The buyers' valuations of the object are independently and uniformly distributed on $[0, 1]$, and each buyer privately knows their own valuation. The seller's value of the object is 0. Suppose the seller adopts the following selling strategy. She approaches one of the buyers (chosen at random) and makes a take-it-or-leave-it offer at a fixed price p_1. If the first buyer accepts the offer, the object is sold to them at the offered price. If the first buyer declines the offer, the seller then approaches the other buyer with a take-it-or-leave-it offer at a fixed price p_2. If the second buyer accepts the offer, the object is sold to them at the offered price. If neither buyer accepts, then the seller keeps the object.

   (a) Derive the optimal values of p_1 and p_2. What is the seller's expected revenue from this selling strategy?

   Solution.

   The seller's profit maximization problem is:

$$\max_{p_1,p_2} \; p_1[1 - F(p_1)] + \{p_2[1 - F(p_2)]\}F(p_1)$$

   We know $F(p) = p$, so the problem is:

$$\max_{p_1,p_2} \; p_1(1 - p_1) + [p_2(1 - p_2)]p_1$$

   The F.O.C.s are:

$$[p_1] : 1 - 2p_1 + p_2(1 - p_1) = 0$$
$$[p_2] : 1 - 2p_2 = 0$$

   Solving these we get $p_1^* = \frac{5}{8}$ and $p_2^* = \frac{1}{2}$. The expected revenue is $\frac{5}{8} \cdot \frac{3}{8} + \frac{5}{8} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{25}{64}$.

(b) Suppose that the seller can also run a first-price auction (without reservation price) instead. Standard theory tells us that in BNE, the buyer's equilibrium strategy is $b_i(v_i) = \frac{v_i}{2}$ and the seller's expected revenue is $\frac{1}{3}$ (This outcome was the answer to the original part (b) of the question). Between this and the outcome in part (a), which one is more efficient? Does the seller always prefer the more efficient option?

Solution.

The seller prefers the sequential offer over the auction because that one maximizes expected revenue. Efficiency wise, the auction is more efficient as the good is always sold whereas the sequential offering case does not lead to the good always being sold.

2. (MSU 2023 Final Q2) Two players jointly own equal shares of an asset and are bargaining to dissolve the joint ownership. Let v_i be i's private value of the whole asset, such that v_1, v_2 are independently drawn from a cdf F_i over the interval $[0, 1]$ (with corresponding pdf f_i). Efficiency requires that i receives the whole asset if $v_i > v_j$. To determine who gets the asset, the two players simultaneously submit bids b_1 and b_2, respectively, and the higher bidder wins the object. In the event of a tie, the winner is determined by a fair coin flip. The winner pays the loser the winning bid amount.

   (a) Suppose bidder j plays a strategy $\sigma_j(v_j)$. What is i's expected payoff from bidding b_i?

   Solution.

   Player i'ex expected payoff from bidding b_i is (Since the cases $v_i = v_j$ are measure 0, we need not include them in the payoff calculation):

$$\begin{aligned} U_i(v_i, b_i \mid \sigma_j(v_j)) =& E[v_i - b_i \mid \sigma_j(v_i) < b_i] \cdot P(\sigma_j(v_j) < b_i) \\ &+ E[\sigma_j(b_j) \mid \sigma_j(v_i) > b_i] \cdot P(\sigma_j(v_j) > b_i) \\ =& \int_{\sigma_j(v_j) < b_i} (v_i - b_i) dF_j(v_j) + \int_{\sigma_j(v_j) > b_i} \sigma_j(v_j) dF_j(v_j) \end{aligned}$$

(b) Let (σ_1, σ_2) be a BNE of the auction. Assume σ_i is strictly increasing and differentiable for $i \in \{1, 2\}$. Describe the pair of differential equations that the strategies must satisfy. [Hint: Choosing to solve the less general case in part (c) may help]

Solution.

Since $v_1, v_2 \overset{iid}{\sim} F([0,1])$, we can rewrite the payoff from (a) as:

$$U_i(v_i, b_i \mid \sigma_j) = (v_i - b_i) F_j\left(\sigma_j^{-1}(b_j)\right) + \int\limits_{v_j > \sigma_j^{-1}(b_i)} \sigma_j(v_j) dF_j(v_j)$$

To solve for the best response for player i, we get the F.O.C. with respect to b_i:

$$\begin{aligned} 0 &= -F_j\left(\sigma_j^{-1}(b_i)\right) + (v_i - b_i) \cdot f_j\left(\sigma_j^{-1}(b_i)\right)\left(\sigma_j'\left(\sigma_j^{-1}(b_i)\right)\right)^{-1} \\ &\quad - \sigma_j\left(\sigma_j^{-1}(b_i)\right) \cdot f_j\left(\sigma_j^{-1}(b_i)\right)\left(\sigma_j'\left(\sigma_j^{-1}(b_i)\right)\right)^{-1} \\ &= -F_j\left(\sigma_j^{-1}(b_i)\right) + \left(v_i - b_i - \sigma_j\left(\sigma_j^{-1}(b_i)\right)\right) \cdot f_j\left(\sigma_j^{-1}(b_i)\right)\left(\sigma_j'\left(\sigma_j^{-1}(b_i)\right)\right)^{-1} \end{aligned}$$

In equilibrium, it must be that $b_i = \sigma_i(v_i)$, so we know

$$\sigma_j'\left(\sigma_j^{-1}(\sigma_i(v_i))\right) \cdot F_j\left(\sigma_j^{-1}(\sigma_i(v_i))\right) = \left(v_i - \sigma_i(v_i) - \sigma_j\left(\sigma_j^{-1}(\sigma_i(v_i))\right)\right) \cdot f_j\left(\sigma_j^{-1}(\sigma_i(v_i))\right)$$

(c) Suppose $v_1, v_2 \overset{iid}{\sim} U[0,1]$. Describe the equations that a symmetric increasing and differentiable bidding strategy must satisfy, and solve the equation. [Hint: Start by conjecturing a functional form $\sigma(v) = c \cdot v$, then solve for c]

Solution.

If using the differential equation approach following from the last part, substituting $F_i(v_i) = v_i$, $f_i = 1$ and $\sigma_1 = \sigma_2 = \tilde{\sigma}$ yields that $\forall v \in [0,1]$,

$$\tilde{\sigma}'(v)v = v - 2\tilde{\sigma}(v) \;\Rightarrow\; c \cdot v = v - 2c \cdot c \;\Rightarrow\; \tilde{\sigma} = \frac{v}{3}$$

Alternatively, we can solve this problem using the expected payoff from (a). Let $\sigma_j(v_j) = c \cdot v_j$, then the expected payoff to i for bidding b_i is:

$$\begin{aligned} &E[v_i - b_i \mid cv_j < b_k] \cdot P(cv_j < b_i) + E[cv_j \mid cv_j > b_i] \cdot P(cv_j > b_i) \\ &= (v_i - b_i)\frac{b_i}{c} + \frac{1}{2}(c + b_i)\frac{c - b_i}{c} \end{aligned}$$

As such, the F.O.C. for i's maximization problem is:

$$\frac{v_i - 2b_i}{c} + \frac{-2b_i}{2c} = 0 \;\Rightarrow\; b_i = \frac{v_i}{3}$$

(d) Prove that the strategy identified in part (c) constitute a symmetric Bayes-Nash equilibrium. [Hint: Check the second order condition]

Solution.

Let $\sigma_j(v_j) = \frac{v_j}{3}$. Then the expected payoff of player I for bidding some other b_i is

$$3(v_i - b_i)b_i + \frac{1 - 9b_i^2}{6}$$

The F.O.C. of this maximization problem is:

$$b_i = \frac{v_i}{3}$$

The second order derivative of the expected payoff is $-9 < 0$. So $b_i = \frac{v_i}{3}$ is indeed a best response, and so this equilibrium is symmetric.

3. (MSU Final 2024) Consider a first-price sealed-bid auction of a single indivisible good with two bidders. Each bidder values the good at either v_L or v_H. The bidders know their own valuation of the good but not the other bidder's valuation. The common prior is that $P(v_i = v_L) = \lambda$. In case of a tie, a fair two-sided coin is flipped to decide the winner of the auction.

   (a) Show that there is no symmetric pure strategy Bayes-Nash equilibrium in this game.

   Solution.
   Suppose that a symmetric equilibrium bidding strategy exists such that $b(v_H) = b_H$ and $b(v_L) = b_L$ such that $b_H > b_L$. First, notice that it must be that $b_L = v_L$. If $b_L < v_L$, the bidders with valuation always has an incentive to bid $b_L = v_L - \varepsilon$ for sufficiently small ε and get payoff $\lambda\varepsilon > 0$.

   Suppose that $b_H < v_H$. For bidders with valuation v_H, their expected payoff for following this strategy is:

$$(1-\lambda)\frac{v_H - b_H}{2} + \lambda(v_H - b_H).$$

   But if they bid $b'_H = b_H + \varepsilon$, for some small ε, instead, their payoff is $v_H - b'_H$. In equilibrium, this must mean:

$$(1-\lambda)\frac{v_H - b_H}{2} + \lambda(v_H - b_H) = v_H - b_h - \varepsilon \Rightarrow \varepsilon = \frac{(1-\lambda)(v_H - b_H)}{2}.$$

   Since b_H is an equilibrium pure strategy, it must mean that no deviation with $\varepsilon \geq 0$ is profitable, so it must be that $b_H = v_H$. However, if $b_H = v_H$, the bidder's expected payoff is 0, meaning they can have profitable deviation by bidding $v_L + \varepsilon < v_H$ and get expected payoff $\lambda(v_H - v_L - \varepsilon)$. In this case, $b_H < v_H$, so there is profitable deviation to bid $b'_H > b_H$ again.

   As such, there is no symmetric pure strategy BNE in this game.

Suppose that, in equilibrium, a bidder of value v_L still bids $b_L = v_L$. Answer parts (b) through (d) to find the symmetric optimal bidding strategy of bidders with v_H.

(b) What is the maximization problem of a bidder of value v_H?

Solution.

$$\pi_i(b_i) = (v_H - b_i)\left[(1-\lambda)P(b_i > b_j) + \lambda\right]$$

So the maximization problem is:

$$\max_{b_i} (v_H - b_i)\left[(1-\lambda)F(b_i) + \lambda\right]$$

(c) Show that bidder i of value v_H is a mixed strategy following the distribution with cdf $F(b_i)$, where $F(b_i)$ satisfies the equality

$$-\int_{v_L}^{b} \frac{d}{db_i}\left[(v_H - b_i)(1 - F(b_i))\right] db_i = \int_{v_L}^{b} \frac{1}{1-\lambda}\, db_i.$$

Solution.

Taking the F.O.C. with respect to b_i of the maximization problem from (b)

$$\begin{aligned}
(v_H - b_i)[(1-\lambda)f(b_i)] - [(1-\lambda)F(b_i) + \lambda] &= 0 \\
\iff (v_H - b_i)f(b_i) - F(b_i) &= \frac{\lambda}{1-\lambda} \\
\iff 1 - F(b_i) + (v_H - b_i)f(b_i) &= \frac{1}{1-\lambda} \\
\iff -\frac{d}{db_i}\left[(v_H - b_i)(1 - F(b_i))\right] &= \frac{1}{1-\lambda} \\
\iff -\int_{v_L}^{b} \frac{d}{db_i}\left[(v_H - b_i)(1 - F(b_i))\right] db_i &= \int_{v_L}^{b} \frac{1}{1-\lambda} db_i
\end{aligned}$$

(d) Solve for $F(b)$ and argue that the support of b is $[v_L, \mu]$ where $\mu \in (v_L, v_H)$. What happens to μ as λ (the probability that the other bidder is type v_L) decreases?

Solution.

From part (b), since $b_L = v_L$, it must be that $b_H > v_L$.

$$
\begin{aligned}
&-\int_{v_L}^{b} [(v_H - b_i)(1 - F(b_i))]' db_i = \int_{v_L}^{b} \frac{1}{1-\lambda} db_i \\
&-[(v_H - b_i)(1 - F(b_i))]\Big|_{v_L}^{b} = \frac{1}{1-\lambda} b_i \Big|_{v_L}^{b} \\
&-[(v_H - b)(1 - F(b))] + [(v_H - v_L)(1 - F(v_L))] = \frac{1}{1-\lambda}(b - v_L) \\
&\Rightarrow F(b) = \frac{\lambda}{1-\lambda} \frac{b - v_L}{v_H - b}
\end{aligned}
$$

Therefore, the high type mixes his bid which follows the CDF of $F(b) = \frac{\lambda}{1-\lambda} \frac{b - v_L}{v_H - b}$. | Clearly, $F(b)$ is undefined at v_H, meaning the support of this CDF must be $[v_L, \mu]$ where $\mu \in (v_L, v_H)$. Specifically, we know that:

$$
\begin{aligned}
F(\mu) = 1 = \frac{\lambda}{1-\lambda} \frac{\mu - v_L}{v_H - \mu} &\iff (1-\lambda)(v_H - \mu) = \lambda(\mu - v_L) \\
\iff \mu = (1-\lambda)v_H + \lambda v_L &\Rightarrow \frac{d}{d\lambda}\mu = -v_H + v_L < 0
\end{aligned}
$$

As λ decreases, μ increases. This means that if the probability that the other bidder's value is v_H increases, the maximum amount a v_H bidder needs to randomize will also increase, to maximize their expected payoff.

4. (MSU Exam 2 2018) Consider a **second-price auction** between two players, where bidder i values the object at v_i, but unlike the standard private value model, assume that the valuations v_1 and v_2 are **correlated**. In particular, assume that each bidder **privately** receives a signal $s_i \in \mathcal{R}$, and value of the good to bidder i is:

$$v_i = \alpha s_i + \beta s_{-i}$$

where $\alpha \geq \beta \geq 0$. (Notice that the standard private value model assumes $\alpha = 1$ and $\beta = 0$). Suppose that both s_1 and s_2 are distributed independently and uniformly over $[0, 1]$. Find a symmetric BNE where each bidder's bid is proportional to her own signal (i.e., in BNE, bidder i bids according to $b_i(s_i) = cs_i$ for some constant $c > 0$. You need to find c).

Solution.

Suppose that in equilibrium, the bidders' strategies are $b_i(s_i) = cs_i$. Bidder i solves the problem:

$$\begin{aligned} &\max_{b_i} \; P(b_i > cs_j) E_{s_j}[\alpha s_i + \beta s_j - cs_j \mid b_i > cs_j] \\ =&\max_{b_i} \; \frac{b_i}{c}\left[\alpha s_i + (\beta - c)\frac{b_i}{2c}\right] \end{aligned}$$

The F.O.C. is:

$$\frac{\alpha s_i}{c} + \frac{\beta - c}{c^2} b_i = 0 \;\Rightarrow\; b_i = \frac{\alpha c}{c - \beta} s_i = cs_i \;\Rightarrow\; c = \alpha + \beta$$

So in symmetric BNE, the bidding strategy is:

$$b_i(s_i) = (\alpha + \beta) s_i$$

5. (MSU 2023 FS Q4) monopolist faces a customer whose valuation for the product is his private information. The monopolist's production cost is 0, and the customer's valuation v is known to be distributed as $U[0, 1]$.

   (a) What is the profit maximizing price for the monopolist? What is the expected profit? Show optimal price and associated profit in the graph drawn above. How much consumer surplus does a customer with valuation v earns under monopoly pricing?

   Solution.

   The demand is the expected value of the transaction. The transaction only happens if $p \leq v$. As such, the monopolist's expected profit is:

$$p \cdot 1 \cdot P(p \leq v) + p \cdot 0 \cdot P(p > v) = p(1 - p)$$

   As such, the profit maximizing price is $p = \frac{1}{2}$ and the expected profit is $\frac{1}{4}$.

(b) Now suppose the customer has an option to fully reveal her valuation to the monopolist. The game proceeds in two steps: first, the customer decides whether or not to reveal v; next, monopolist offers a price based on the customer's disclosure decision. Note that in any equilibrium of this game, the monopolist charges a price $p = v$ to a customer who discloses her valuation, and an uniform price p_0 (say) to all customer who does not disclose her type. What is the minimum value of p_0 that can be sustained in an equilibrium of this game? Can the customer be better off by having such an disclosure option?

Solution.

In EQ, if the customer reveals v, then the price is $p = v$.

In EQ, if the customer does not reveal v, then the price is some p_0.

The merchant either sees some realized $\bar{v} < v_0$ or an unknown $v > v_0$. When the merchant sees $v \geq v_0$, their expected profit for charging p is:

$$p \cdot P(v \geq p \mid v \geq v_0) = p \cdot \frac{1-p}{1-v_0}$$

The F.O.C. is:

$$\frac{1-2p}{1-v_0} = 0 \;\Rightarrow\; p_0 = \frac{1}{2}$$

Next, notice that if $p_0 = \frac{1}{2}$, it is in the customer's best interest to disclose v if $v \leq \frac{1}{2}$. This means that $v_0 = \frac{1}{2}$.

You might think that the customer is better off in this case. However, if $v \leq \frac{1}{2}$, the customer gets charged $p = v$, meaning they still have 0 surplus. If $v > \frac{1}{2}$, then they get charged $\frac{1}{2}$, which is the same as before. So the customer cannot be better off by having this disclosure option.

6. (MSU 2020 Exam 3 Q1) Each person may have a privately known pre-existing health condition with the severity of the condition uniformly distributed between 0 and 1 in the population. Any person without a health insurance gets a payoff 0. A person with condition severity (type) θ will derive a payoff θ from buying a health insurance. The health insurance can only charge a uniform price p to all individuals, but the cost of insuring each individual is different: The cost of insuring a type θ individual is $\theta/2$.

   (a) What is the profit function for a monopoly health insurance provider?

   Solution.
   When the price is p, individuals with $\theta > p$ are willing to participate in the insurance. The total revenue is $p(1-p)$. The total cost of having individuals $\theta \in [p, 1]$ participate is

$$c(p) = \int_p^1 \theta/2 d\theta = \theta^2/4\Big|_p^1 = \frac{1}{4}(1-p^2).$$

   Profit from price p is

$$\pi(p) = p(1-p) - \frac{1}{4}(1-p^2) = \frac{1}{4}\left[4p - 3p^2 - 1\right].$$

   (b) What is the profit-maximizing price when there is a single health insurance provider?

   Solution.
   When the price is p, individuals with $\theta > p$ are willing to participate in the insurance. The total revenue is $p(1-p)$. The total cost of having individuals $\theta \in [p, 1]$ participate is

$$c(p) = \int_p^1 \theta/2 d\theta = \theta^2/4\Big|_p^1 = \frac{1}{4}(1-p^2).$$

   Profit from price p is

$$\pi(p) = p(1-p) - \frac{1}{4}(1-p^2) = \frac{1}{4}\left[4p - 3p^2 - 1\right].$$

The FOC is

$$4 - 6p^* = 0 \Rightarrow p^* = 2/3.$$

(c) Suppose two health insurance providers simultaneously announce their prices, and individuals choose to buy from one insurer. What is the equilibrium price?

Solution.

When there are two insurers, they compete and drive their profit to zero. The profit is zero when

$$\frac{1}{4}\left[4p - 3p^2 - 1\right] = -\frac{1}{4}(3p-1)(p-1) = 0.$$

The solution is $p = 1$ and $p = 1/3$. The price $p = 1$ is not sustainable, so $p = 1/3$ is the unique equilibrium price when there are two insurers.

(d) Suppose a law penalizes any individual who does not buy a health insurance by incurring a cost of 1/4. (The penalty becomes government revenue). What is the profit-maximizing price when there is a single health insurance provider?

Solution.

Given price p, a type-θ individual is willing to pay p if and only if

$$\theta - p \geq -1/4,$$

so $\theta \geq p - 1/4$. For any price $p > 1/4$, individuals buying the insurance are of type $\theta \in [p - 1/4, 1]$. The revenue is $p(1 - (p - 1/4)) = p(5/4 - p)$. The cost is

$$c(p) = \int_{p-\frac{1}{4}}^{1} \theta/2 d\theta = \theta^2/4\Big|_{p-\frac{1}{4}}^{1} = \frac{1}{4}(\frac{15}{16} - p^2 + \frac{1}{2}p).$$

The profit for any $p \geq 1/4$ is

$$\pi(p) = p\left(\frac{5}{4} - p\right) - \frac{1}{4}(\frac{15}{16} - p^2 + \frac{1}{2}p)$$

FOC is $5 - 6p - \frac{1}{2} = 0$ and solving for p yields $p^* = 3/4$.

3 Go the Extra Mile

1. (MSU 2017 Exam 1 Q3) Suppose that there are n bidders bidding for an object in a sealed-bid all-pay auction. That is, players submit their bids simultaneously; only the highest bidder is awarded the object, but each player must pay his own bid irrespective of whether he wins or not. Bidder's values are drawn independently from the uniform distribution on $[0;1]$ and are privately known.

   (a) Find a symmetric Bayesian Nash equilibrium of this game in monotonically increasing bidding strategies. Calculate the seller's expected profit.

   Solution.
   The bidder's problem is
   $$\max_b \ \left[b^{-1}(b)\right]^{n-1} v - b.$$
   The FOC is
   $$(n-1)\left[b^{-1}(b)\right]^{n-2} v \frac{1}{b'(v)} - 1 = 0,$$
   or,
   $$(n-1)\, v^{n-1} = b'(v).$$
   So,
   $$b(v) = \frac{n-1}{n} v^n.$$
   The seller's expected revenue is:
   $$n \int_0^1 b(v)\, dv = \frac{n-1}{n+1}.$$

   (b) Now suppose the object has a common value: all bidders value it at 1. The auction rule is still the same as above. Find a symmetric Bayesian Nash equilibrium of this game. Calculate the seller's expected profit. [Hint: Note that this is a game of *complete information*. Also, first argue that a pure strategy NE cannot exist and then try to compute a mixed strategy NE where a strategy is simply a probability distribution over $[0;1]$.]

   Solution.
   No pure strategy NE exists. A sketch of the proof is as follows: For any strategy profile $(b_1^*, ..., b_n^*)$ such that $b_1^* \leq ... \leq b_n^*$ (we can always relabel the players to

ensure this ordering) at least one losing bidder would deviate to 0 or the winning bidder can reduce her bid. If it is a tie at the top, one of the winners would also deviate to a lower bid.

Now, consider a symmetric mixed strategy where every player randomizes over bids $b \in [0,1]$ according to the density $F(v) = \Pr(b \leq v)$, $v \in [0,1]$. All bids on the support gives same payoff, say v^*. So, by bidding b a player gets:

$$F(b)^{n-1} \times 1 - b = v^*.$$

So, $F(b) = (b + v^*)^{\frac{1}{n-1}}$. But $F(1) = 1$. So, $v^* = 0$. Hence, a symmetric mixed strategy NE is one where all players bid according to $F(b) = b^{\frac{1}{n-1}}$. The expected revenue of the seller is:

$$n \int_0^1 b dF(b) = n \int_0^1 \frac{1}{n-1} b^{1/(n-1)} db = 1.$$

2. (MSU 2021 Exam 1 Q2) Consider two firms competing in Stakelberg fashion. Firm 1 is the leader who chooses output q_1. Firm 2 observes q_1 before choosing its output q_2. Fixed costs are 0 to both firms. Firm 1 has constant marginal cost $c_1 > 0$, and this is known to both firms. Firm 2 might be a low-cost firm, given by constant marginal cost $c_L > 0$, or a high-cost firm, given by constant marginal cost $c_H > 0$. Firm 2 knows its type, but firm 1 only knows that firm 2 is a low-cost firm with probability $\frac{1}{2}$. Suppose inverse market demand is given by $p = 100 - Q$ where Q is the total output ($Q = q_1 + q_2$) in the market. Suppose $c_1 = 6, c_L = 4$, and $c_H = 12$.

   (a) Formulate this situation as a Bayesian game.

   Solution.
   Bayesian game Γ_N is defined by $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}\Theta, F(\cdot)]$ where

$$\begin{array}{lcl} I & = & \{1,2\} \\ S_i & = & \mathbb{R}_+ \text{ for } i = 1,2 \\ u_i(q_1, q_2, c_i) & = & (100 - q_1 - q_2)q_i - c_i q_i \\ \Theta = \Theta_1 \times \Theta_2 & = & \{c_1\} \times \{c_L, c_H\} \\ F & = & \left\{ \begin{array}{lcl} P(c_2 = c_L) & = & 1/2 \\ P(c_2 = c_H) & = & 1/2 \end{array} \right. \end{array}$$

(b) Find a Bayesian Nash Equilibrium. [Note: Remember the distinction between equilibrium strategies and outcomes.]

Solution.

Start from firm 2.

Low-cost firm's expected payoff: $(100 - q_1 - q_2)q_2 - c_L q_2$

Using FOC, we have $q_{2L}(q_1) = \dfrac{100 - q_1 - c_L}{2} = 48 - \dfrac{q_1}{2}$

Low-cost firm's expected payoff: $(100 - q_1 - q_2)q_2 - c_H q_2$

Using FOC, we have $q_{2H}(q_1) = \dfrac{100 - q_1 - c_H}{2} = 44 - \dfrac{q_1}{2}$

Firm 1's expected payoff: $\frac{1}{2}[(100-q_1-q_{2L})q_1 - c_1 q_1] + \frac{1}{2}[(100-q_1-q_{2H})q_1 - c_1 q_1]$

Using FOC, we have $q_1 = 50 + \dfrac{c_H + c_L}{4} - c_1 = 48$

Then Bayesian NE is $(48, \max\{48 - \frac{q_1}{2}, 0\}$ if $c_2 = c_L, \max\{44 - \frac{q_1}{2}, 0\}$ if $c_2 = c_H)$.

(c) Compute the equilibrium profit for each firm.

Solution.

In the equilibrium, $q_1 = 48, q_{2L} = 24$ and $q_{2H} = 20$.

Then market price is 28 if $c_2 = c_L$ and 32 if $c_2 = c_H$.

Using it, profit is

$$\pi_1 = \begin{cases} 48 \times (28 - 6) = 1056 & \text{if} \quad c_2 = c_L \\ 48 \times (32 - 6) = 1248 & \text{if} \quad c_2 = c_H \end{cases}$$

$$\pi_2 = \begin{cases} 24 \times (28 - 4) = 576 & \text{if} \quad c_2 = c_L \\ 20 \times (32 - 12) = 400 & \text{if} \quad c_2 = c_H \end{cases}$$

3. (MSU 2016 Exam 2 Q3) Consider the following model of trade under adverse selection. A seller S wants to sell a product to a buyer B. The product's quality $q \in [0, 1]$ is known to S but not to B. Assume that the buyer's prior belief about q is uniform on $[0, 1]$. When the product quality is q, its value to S is $q/3$ whereas its value to B is exactly q.

(a) Show that no quality gets sold (expect for 0) in the competitive equilibrium with price zero (where price is equal to the expected quality available on the market).

Solution.

There are two competitive equilibria, one with $p = 0$ and other with $p = 1/2$. Note that

$$\mathbb{E}\left(q \mid q/3 < p\right) = \begin{cases} \frac{3}{2}p & if\ p < \frac{1}{3} \\ \frac{1}{2} & otherwise \end{cases}.$$

So, the equation

$$p = \mathbb{E}\left(q \mid q/3 < p\right)$$

has two solutions: 0 and $\frac{1}{2}$. There is no trade in the former equilibrium but all types are traded in the latter one.

(b) Now consider the following trading rule: For every qualily $q \in Q$, S posts a take-it-or-leave-it price p and may choose to report the true quality. That is, given quality $q \in Q$, S sends a message $m \in \{q, \emptyset\}$ where $m = \emptyset$ means S chooses not to reveal the quality. Upon observing $(\bar{p}, m)$, B decides whether to buy or not at price $\bar{p}$. The seller's payoff is $p - q/3$ and the buyer's payoff is $q - p$. Assume that the buyer buys the product as long as p does not exceed the product's expected quality. Show that there exists a PBE where S always reveals the true quality and all qualities are traded at $p = q$. Clearly state the strategies of S and B and the associated belief of the buyer. [*Hint:* The off-equilibrium belief must be such that upon observing $m = \emptyset$, B forms a very unfavorable opinion about the underlying quality.]

Solution.

Consider the belief $\Pr\left(q = 0 \mid m = \emptyset\right) = 1$. Now, S revealing q setting $p = q$ and B always buying is PBE. All $m \in [0, 1]$ is on-eqm, and hence, the belief is $\Pr\left(q = q' \mid m = q'\right) = 1$.

(c) Consider a further variation of the model described so far. Suppose that S is informed about the quality with a probability $\alpha < \frac{1}{2}$ and he cannot report q to B. What is the competitive equilibrium price in this situation and what quality levels, if any, are traded at equilibrium? (You should be able to derive the subset of qualities that are traded in equilibrium.)

Solution.

The equilibrium p the expected quality of the product given q. Now, if $p < 1/6$, a seller who does not know her quality would not be on the market since her expected valuation of the product is $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$. Hence, the expected quality of

the product is:

$$\begin{cases} \mathbb{E}\left(q \mid q/3 < p\right) & if\ p < \frac{1}{6} \\ (1-\alpha)\frac{1}{2} + \alpha\mathbb{E}\left(q \mid q/3 < p\right) & otherwise \end{cases},$$

or

$$\begin{cases} \frac{3}{2}p & if\ p < \frac{1}{6} \\ (1-\alpha)\frac{1}{2} + \alpha\frac{3}{2}p & if\ \frac{1}{6} \leq p \leq \frac{1}{3} \\ \frac{1}{2} & otherwise \end{cases}.$$

The solutions are $p = 0$ and $p = 1/2$. So, all qualities are traded in equilibrium with $p = 1/2$ and none is traded in equilibrium with $p = 0$.